The Monge point of a tetrahedron in descriptive geometry

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The Monge point of a tetrahedron is the equivalent, in three dimensions, of the orthocenter of a triangle [1, 2]. It was defined by Gaspard Monge (1746-1818), a revolutionary spirit whose work covered numerous areas of mathematics, physics and chemistry [3]. In mathematics, while he wrote important papers on differential geometry, his best known contribution is descriptive geometry [4], a simple method that he designed when he was in his early twenties to describe buildings and objects in three dimensions using projections on two perpendicular planes: a map view and a side view, also called elevation. While all students in the XIXth and XXth centuries practiced this method extensively, as engineering drawing, it tends to be forgotten nowadays. It is a pity because it contains numerous subtleties to describe surfaces and their intersections, and it is also, therefore, a powerful educational method to train the brain in three dimensions. Gaspard Monge is also known as the scientific mastermind, from 1798 to 1799, behind the Egypt adventure of his friend and pupil, general Napoleon Bonaparte. From Egypt, like many of the members of the expedition, Monge brought back a fascination of Egyptian design and, in particular, the pyramid. However, it is only a few years later, around 1809, when finding leisure from his numerous duties in the imperial system, that he did consider the geometrical properties of the tetrahedron, referred to as the irregular triangular pyramid, and defined the particular point now known as the Monge point [5]. At about the same time, Lazare Carnot (1753-1823) had written on the generalization to three-dimensions of the analytical trigonometric relations of the triangle, starting with the case of the triangular pyramid [6].

Let us consider an irregular tetrahedron with base ABC and summit[1] D, shown from above in *figure 1*. For each edge, say AB, the Monge plane Π_{AB} is defined as the plane perpendicular to the edge passing through the midpoint of the opposite edge, CD for edge AB. For one face, for example the base ABC, the three Monge place are concurrent. Indeed, the plane traces coincides with the heights of the mid-height triangle A'B'C', which intersect at the orthocenter of this triangle. The intersection of the three Monge planes of a given face is called the Monge normal. The four Monge normals of the tetrahedron concur at the Monge point M [1, 5]. Another property of the point M is that the centroid G is the middle of M and the center O of the circumscribed sphere, thus:

$$\overrightarrow{OM} = 2\overrightarrow{OG} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}\right).$$

This relation, alternatively, can be used to define M. Indeed, let us consider the projection on the base plane (*figure 1*), using the same symbols for the actual points and their projections on the figure plane. In this plane, consider the vector $\overrightarrow{O'M}$, where O' is the incenter of the mid-height triangle A'B'C'. We have:

[1] following traditional vocabulary, not necessarily in use now in the English language literature

$$\overrightarrow{O'M} = \overrightarrow{O'O} + \overrightarrow{OM} = \overrightarrow{O'O} + \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}\right).$$

However, points O', A', B', and C' are the midpoints of OD, AD, BD and CD, respectively, giving:

$$\overrightarrow{O'M} = \overrightarrow{O'O} + \frac{1}{2}\left(2\overrightarrow{O'A'} + 2\overrightarrow{O'B'} + 2\overrightarrow{O'C'} + 2\overrightarrow{OO'}\right) = \overrightarrow{O'A'} + \overrightarrow{O'B'} + \overrightarrow{O'C'},$$

which means [7, p8] that the projection of M on the base is the orthocenter of triangle A'B'C', in agreement with our first definition of M above. Since this is true for each face of the tetrahedron, it means that the point M defined by the vectorial relation $\overrightarrow{OM} = \left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}\right)/2$ is at the intersection of the 4 Monge normals, which also proves, incidentally, that these 4 Monge normals do intersect at one point.

To our knowledge, Monge himself did not write down a construction of the Monge point using descriptive geometry. It is unlikely, however, that he would not have done the exercise during his celebrated classes at Ecole Polytechnique, which was his main effort at the time. In the following, we present two constructions, which we consider as enlightening geometry drills, using ruler, square rule and compass, and the methods of descriptive geometry.

The first construction (*figure 2*) uses the definition of the Monge point by the intersection of two Monge normals. The view from above is presented on the left side. Without loss of generality, the BC edge of the base is taken as projection axis to obtain the elevation shown on the right. Following Monge conventions [4], a point shown using a capital letter in the map view is indicated with a small letter in the elevation. Unlike Monge, however, we indicate the points constructed in turn by numbers in increasing order. The first step is to construct the triangle at mid-height 1,2,3 on the left side, which already gives the Monge point M as the orthocentre of this triangle, then place the corresponding points 4 and 5 in the elevation. The horizontal line drawn from M and extended into the elevation gives a first Monge normal. A second Monge normal must be found to obtain m. Returning to the map view, we concentrate on another mid-height triangle 1,6,7 erected from base BCD. The trace of this second mid-height triangle is obtained from 8 to 4 in the elevation. To find the orthocentre of the second mid-height triangle, a trick often used by Monge is used [4]: using the line 6,7 as a hinge, hence point 8 in the elevation, the mid-triangle is rotated to the horizontal plane, abc, giving point 9. Drawing a horizontal line from 9, we obtain point 10 on the left side. Now, the triangle 6,7,10 on this view is the second mid-height triangle flapped onto the base ABC. The orthocentre 11 of this triangle is easily obtained, which then, drawing a horizontal line, gives point 12 in the elevation. Rotating back the triangle in its real position, point 13 is found. The perpendicular line to 4,8 in 13 then intersects the first Monge normal in m.

A second construction (*figure 3*) can be proposed using the property of the Monge point in relation to the centroid G and the circumcenter O of the tetrahedron, using again the trick of hinge rotations of accessory planes. As in the previous construction, the points of the mid-height triangle from base ABC (points 1 to 5) are placed, giving the position of M in the map view as the orthocentre. Then, here, the idea is to first place the centroid, then the circumcenter. The centroid is easily found as the intersection of opposite medians (*figure 3*), such as 2,6 and 3,7, giving G on the left side. Then intersection of 4,b and 5,9 gives g in the elevation. To find the circumcenter, we use a construction similar to the one given in his early years by Monge [4]. The center O of the circumscribed sphere is, in the map view, the circumcenter O of triangle ABC, which is obtained from the intersection of two perpendicular lines erected from 7 and 8. The horizontal line drawn from O and extended in the elevation gives the line on which lies o, cutting the base in point 9. Let us now consider, in the map view, the vertical plane containing O and C. If this plane is now shown in the elevation, the distance OC from point 9 gives point 10. Similarly, in the map view, the vertical plane containing O and D, can be shown in the elevation, which the distance OD from point 9 gives point 11. Drawing a vertical line from D in the elevation, we obtain point 12. Triangle 9,11, 12 represents, flapped onto the elevation, the triangle seen from above as OD in the map view. Since O (o) is the centre of the circumscribed sphere, the distance from o to 9 and from o to 12 must be the same, equal to the radius of the circumscribed sphere. The triangle o,10,12 is therefore isosceles, and point o can simply be obtained as the intersection between the horizontal line passing by 9 and the normal of segment 10,12 passing by their mid-point 13, easily found by a compass construction. Once o is found, the Monge point m is obtained as the symmetric from o across g, or as the intersection of line og with the horizontal line obtained from M in the map view.

While the second construction is slightly more complicated, it offers a nice alternative, with exercising the relative positions of the centroid G, the circumcenter O and the Monge point. With these simple constructions, we also see how the two perpendicular views are used in descriptive geometry, with incessant playing back and forth between the two views, getting points logically one after the other, in a perfectly rigorous manner. The geometric construction by hand also offers a powerful tool complementary to analytical vectorial geometry, which can be very efficient, but lacks a feeling on the relative positions of the points within the objects. Both approaches should be taught to the students, which can be more or less talented in one or the other, but can slowly but surely gained confidence in both, with, in addition, the special feeling of satisfaction and contemplation specific to geometry.

References

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Figure 1

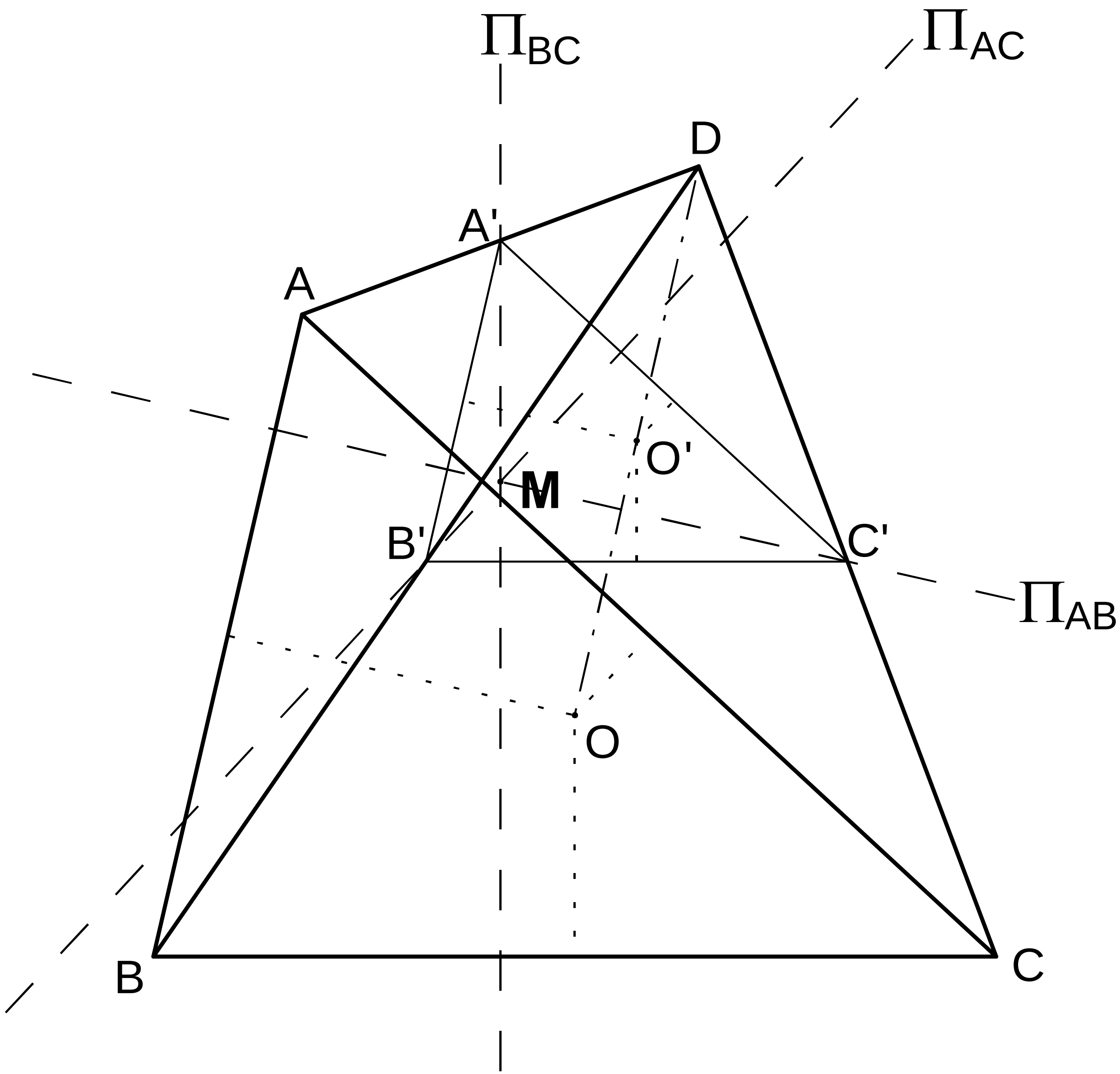


Figure 2

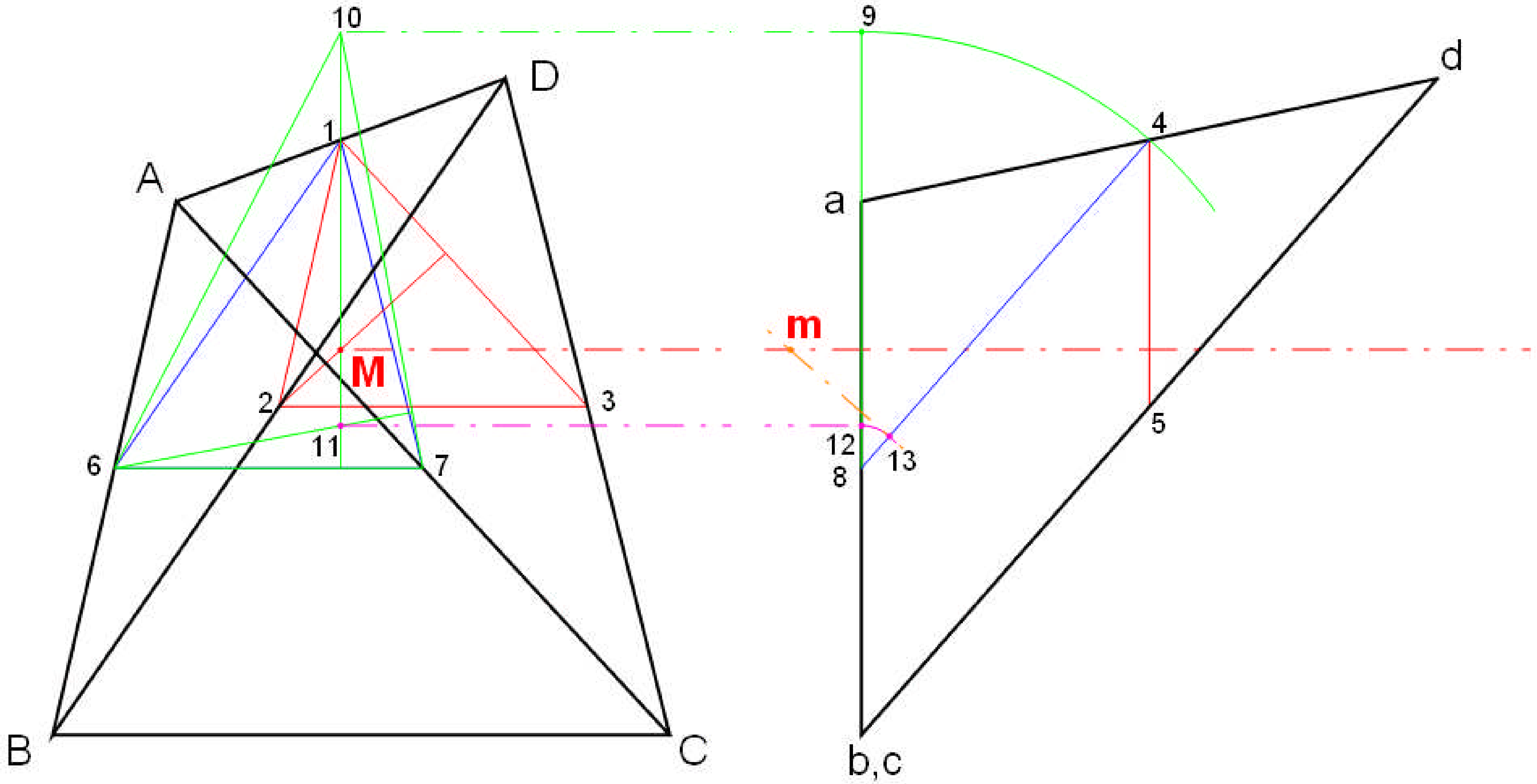
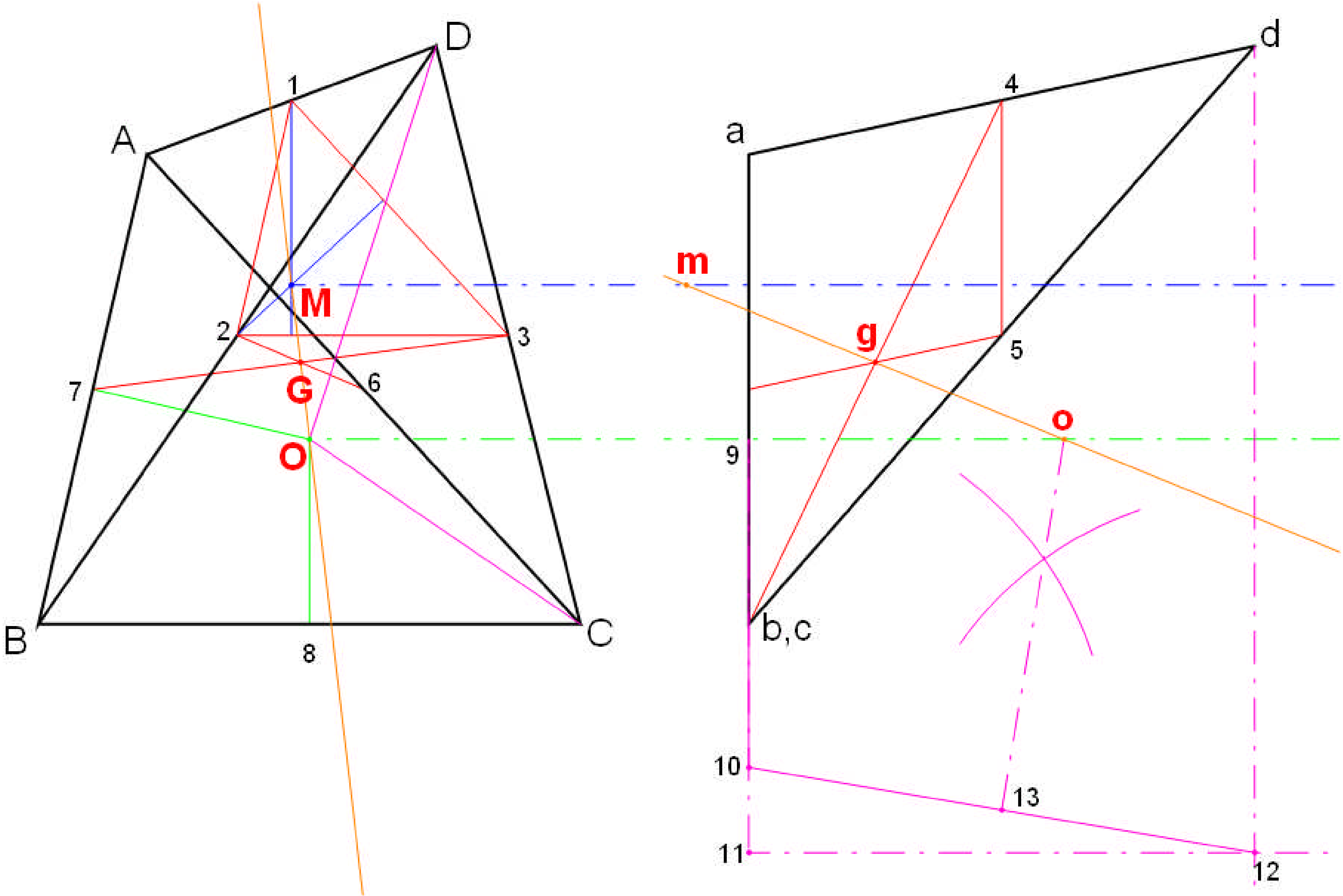


Figure 3



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