NON-EXTENSIVE MODEL FOR TURBULENT FLOW IN A RAPIDLY ROTATING ANNULUS *

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Abstract

We have conducted experiments on turbulence in a rapidly rotating annular tank. The flow is driven by pumping fluid into the tank through a semi-circle of holes; the fluid flows out through a semi-circle of holes on the opposite side of the annulus. We model the system using statistical mechanics with a non-extensive entropy. Assuming conservation of potential enstrophy and energy, we deduce a mean vorticity profile that agrees with the observations. Further, we find that the probability distribution for the vorticity is better fit by nonextensive than extensive theory.

Equilibrium statistical mechanics has long been used to model two-dimensional (2D) turbulence, but 2D turbulent flows can at best only be approximated in real systems [1]. Thin electrolytic layers and soap films [2] are used as approximately 2D systems but are limited by 3D dissipation and low Reynolds number. We use rapid rotation to achieve quasi-2D flow (Rossby number = 5.3×10^{-2}), and strong pumping of fluid produces small-scale vortices that drive a turbulent flow (Reynolds number = 7000). Further, the annulus is tall so that the Ekman dissipation in the top and bottom boundary layers is small (Ekman number = 5×10^{-4}); that is, vortex turnover time is short compared to the Ekman time. Measurements of the vorticity field in this system were reported in [3].

Here we examine the applicability of nonextensive statistical mechanics to our turbulent flow. The flow contains coherent vortices with sizes

*This work is supported by O.N.R.

ranging up to the size of the system. The presence of large coherent structures indicates long range correlations, which suggests that a nonextensive theory would be more appropriate than the usual Boltzmann-Gibbs extensive statistical theory. Our analysis uses the Tsallis nonextensive generalization of entropy in statistical mechanics [4]. This approach has proved to be useful in describing the statistics of turbulence [5]. [1] We propose a new model by following the statistical approach of Miller[1], who obtained relationships for the measurable ("dressed") vorticity in turbulent 2D flow from consideration of the "microscopic vorticity" [6].

Our experiments were conducted on flow in an annular tank described in [3]. [2]. Particle Image Velocimetry (PIV) is used to obtain the full 2D velocity field.

We have found that maximizing either the extensive or the nonextensive entropy[6] leads to the same expression relating the streamfunction and the vorticity, $\omega = -\nabla^2\psi$:

$$\nabla^2\psi - (\beta_{\mathrm{Ross}} r + \frac{\beta}{2\gamma}\psi) = 0, \tag{1}$$

which shows a linear relationship between ψ and ω. The Rossby parameter β_{Ross} is determined by the geometry of the system, and the parameter $\frac{\beta}{2\gamma}$ is obtained by fitting Eq. 1 to the streamfunction ψ from the experiment, using appropriate boundary conditions. [3]

We have derived expressions for Miller's dressed vorticity[1] using the conservation of energy and potential enstrophy in both the extensive and nonextensive statistical approaches,

$$p^{\mathrm{ext}}(\omega) \propto \int d\vec{r} \frac{1}{\phi(\vec{r})} e^{-\gamma\phi(\vec{r})^2} \sinh(2\gamma\alpha\phi(\vec{r})) \tag{2}$$

$$\begin{aligned} p^{\mathrm{nonext}}(\omega) \propto & \int d\vec{r} \frac{f(\omega)}{\phi(\vec{r})} [1 - \frac{(1-q)\gamma}{f(\omega)}\phi(\vec{r})^2]^{\frac{q}{1-q}} \{[1 + 2\frac{(1-q)\gamma\alpha}{f(\omega) - (1-q)\gamma\phi(\vec{r})^2}\phi]^{\frac{1}{1-q}} \\ & -[1 - 2\frac{(1-q)\gamma\alpha}{f(\omega) - (1-q)\gamma\phi^2}\phi(\vec{r})]^{\frac{1}{1-q}}\} \end{aligned} \tag{3}$$

where, as shown in [6], $\phi(\vec{r}) := \omega + \beta_{\mathrm{Ross}} r + \frac{\beta}{2\gamma}\psi(\vec{r})$, α is the fluctuation limit of the microscopic vorticity and γ is the Lagrange multiplier in front of the potential enstrophy.

The experimental result for the vorticity, averaged in the azimuthal direction, is compared in Fig. 1 with the prediction that we have obtained from nonextensive statistical mechanics. The two curves have the same qualitative features, the difference arising in part from the assumption of an inviscid fluid in the model. The inviscid model cannot

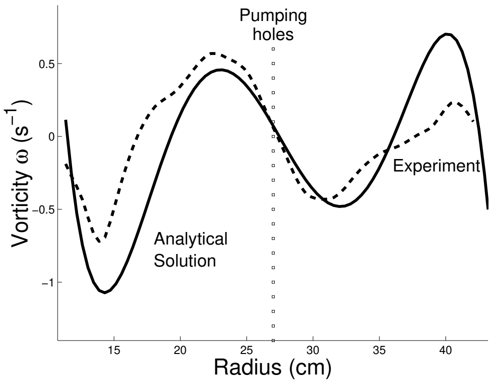

Figure 1. A comparison of the predicted radial dependence of azimuthally averaged vorticity (solid line, Eq. 1 with $\frac{\beta}{2\gamma} = -0.158$) with measurements for a 2.5 Hz rotation rate and $= 150 \ \ \mathrm{cm}^3/\mathrm{s}$ pumping rate.

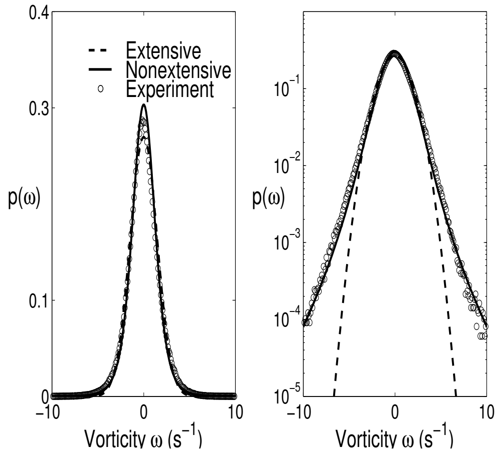

Figure 2. A comparison of the predicted PDF for the vorticity given by extensive theory (Eq. 1, dashed curves) and nonextensive theory (Eq. 2, solid curves) with the experimental data (2.5 Hz rotation rate and $=$ $150 \ \ \mathrm{cm}^3/\mathrm{s}$ pumping rate). A linear plot of the PDF is shown on the left to emphasize the fit near the peak, and a log plot is shown on the right to emphasize the fit in the tails.

capture a prominent feature of the data, the production of vortices at the inner and outer walls; these vortices certainly must affect the vorticity distribution in the interior of the annulus.

The PDF for the vorticity predicted by our extensive and nonextensive analyses is compared with our measurements in Fig. 2. The parameter values obtained in a least-squares fit to the nonextensive model are are $q = 1.9 \pm 0.2$, $\gamma = 0.146 \pm 0.006$ and $\alpha = 0.7 \pm 0.2$; for the extensive model, $\gamma = 0.25 \pm 0.03$ and $\alpha = 0.7 \pm 0.2$. The nonextensive model fits the data well over the entire range including both the peaks and the tails of the PDF. The broad tails of the distribution arise because of the large vortices that form when small vortices of like sign merge.

In summary, we have found that the assumptions of energy and potential enstrophy conservation in a rapidly rotating inviscid flow lead in nonextensive statistical theory to predictions for the vorticity PDF and the radial dependence of the vorticity that agree well with our measurements on turbulent flow in a rapidly rotating annular tank. The nonextensive theory yields broad tails of the PDF, which are not explained by extensive theory.

Notes

1. For 3D turbulence, the assumption that an effective energy is conserved leads to a probability distribution functions (PDF) for velocity differences that fit data over a wide range of length scales. However, the assumption of conserved effective energy is not valid in a rotating fluid system.

2. The tank has an inner radius ($r_i = 10.8$ cm), outer radius ($r_o = 43.2$ cm), a sloping bottom and is covered by a solid transparent lid. The tank is spun with rotation frequencies $\Omega/2\pi$ up to 2.5 Hz. The bottom depth varies from 17.1 cm at the inner radius to 20.3 cm at the outer radius. Water is continuously pumped through the tank in closed circuit via a ring of 120 circular holes located at the bottom of the tank. The overall pumping rate is between 150 and 350 cm^3/s. The forcing holes are located at the mean radius of the annulus, $r_f = 27$ cm. In order that zonal flow is not directly forced, the forcing is arranged in two semi-circles: one semi-circle of the forcing ring contains sources and the opposite semi-circle contains sinks.

3. The condition $\int \omega|_{r=r_f} d\theta = 0$ holds since the forcing holes act as sources in one semi-circle and sinks in the opposite semi-circle on a ring at $r = r_f$. The second boundary condition is the conservation of the total circulation, $\oint \vec{u} \cdot d\vec{l} = 0 = \int d\vec{r}\omega$.

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