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Normal modes of the Earth

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Abstract. The free oscillations of the Earth were observed for the first time in the 1960s. They can be divided into spheroidal modes and toroidal modes, which are characterized by three quantum numbers n, l, and m. In a spherically symmetric Earth, the modes are degenerate in m, but the influence of rotation and lateral heterogeneities within the Earth splits the modes and lifts this degeneracy. The occurrence of the Great Sumatra-Andaman earthquake on 24 December 2004 provided unprecedented high-quality seismic data recorded by the broadband stations of the FDSN (Federation of Digital Seismograph Networks). For the first time, it has been possible to observe a very large collection of split modes, not only spheroidal modes but also toroidal modes.

1. Introduction

Seismic waves can be generated by different kinds of sources (tectonic, volcanic, oceanic, atmospheric, cryospheric, or human activity). They are recorded by seismometers in a very broad frequency band. Modern broadband seismometers which equip global seismic networks (such as GEOSCOPE or IRIS/GSN) record seismic waves between 0.1 mHz and 10 Hz. Most seismologists use seismic records at frequencies larger than 10 mHz (e.g. [1]). However, the very low frequency range (below 10 mHz) has also been used extensively over the last 40 years and provides unvaluable information on the whole Earth. In this frequency range, free oscillations of the Earth can be observed. Normal modes of free oscillation were predicted before being observed. In 1882, Horace Lamb provided the first comprehensive mathematical treatment of the free oscillations of a non-gravitating sphere, distinguishing spheroidal and toroidal oscillations. Free oscillations of the Earth result from the constructive interference of traveling waves in opposite directions. They are characterized by an eigenfrequency and an eigenfunction. Alterman et al. [2] calculated the theoretical values of the eigenfrequencies for the gravest modes in a spherically symmetric Earth model. These were first observed unambiguously following the great Chile earthquake of 1960 by Benioff et al. [3]. Each mode, either spheroidal or toroidal, is characterized by three quantum numbers, n, l, and m. In a spherically symmetric Earth, the modes are degenerate in m (degenerate multiplets) and can be sorted according to n and l. Normal modes have been used since the 1960s for deriving the radial variations of the isotropic physical quantities characterizing the structure of the Earth (density $\rho(r)$, P-wave velocity $V_P(r)$ and S-wave velocity $V_S(r)$), such as in models 1066A and 1066B [4]. Since normal mode eigenfunctions constitute a complete basis for the Earth displacement, they are used to calculate synthetics seismograms (e.g. [5]). A remarkable application of normal mode theory is the inversion of the centroid moment tensor for large earthquakes [6]. The different singlets are usually mixed into one multiplet, a complex combination of all singlets. The effects of the earth rotation and lateral heterogeneities split the degeneracy of the eigenmodes. The occurence of the Great Sumatra-Andaman earthquake (Magnitude Mw$\simeq$ 9.3) is a unique opportunity to reconsider the observations of split modes and to considerably extend the dataset of singlets. For the gravest modes, individual singlets are very well separated.

2. Basic normal mode theory

In the solid elastic Earth, the basic equation which governs the displacement $\mathbf{u}(\mathbf{r}, t)$ is the elasto-dynamic equation [7]:

$$\rho(\partial_t^2 + H)\mathbf{u}(\mathbf{r}, t) = \mathbf{f}(\mathbf{r}, t), \tag{1}$$

where H is an integro-differential operator and $\mathbf{f}$ represents all internal forces. We assume that $\mathbf{f}$ is equal to 0 for $t < 0$. The solution to the above equation, $\mathbf{u}(\mathbf{r}, t)$, can be expanded in terms of a complete set of vector functions, commonly called normal modes.

Let us consider a particular reference Earth model with operator H_0 and model density ρ_0. The normal modes $\mathbf{u}_k(\mathbf{r})$ in this reference model are such that

$$H_0\mathbf{u}_k(\mathbf{r}) = \omega_k^2\,\mathbf{u}_k(\mathbf{r}), \tag{2}$$

where ω_k are the eigenfrequencies. The index k denotes a particular set of three quantum numbers n (radial order), l (spherical harmonic degree), and m (azimuthal order). These eigenmodes form a complete orthogonal basis. The reader is referred to Aki & Richards [8], Woodhouse [5] or Dahlen & Tromp [9] for the fundamentals of normal mode theory.

For an elastic spherically symmetric earth model, two independent families of modes exist: the spheroidal modes resulting from the constructive interference of Rayleigh waves and the coupling between P and SV waves, and the toroidal modes including Love and SH waves. The eigenvector $\mathbf{u}_k$ depends on all three quantum numbers n, l and m, where m is the azimuthal order such that $-l \leq m \leq l$. In the case of a Spherically symmetric Non-Rotating Elastic Isotropic (SNREI) reference Earth model, however, the eigenfrequency ω_k depends on the two quantum numbers n and l only; the energy level k is said to be degenerate, with a degree of degeneracy $g_k = 2l + 1$. The eigenfunctions are orthogonal and normalized according to

$$\int_{\oplus} \rho_0\,\mathbf{u}_k^* \cdot \mathbf{u}_{k'}\,dV = \delta_{kk'}. \tag{3}$$

In a spherical-polar coordinate system (r, θ, ϕ), the displacement $\mathbf{u}_k$ in the reference SNREI can be expressed in terms of three radial eigenfunctions U_{nl}, V_{nl}, and W_{nl}:

$$\mathbf{u}_k(\mathbf{r}) = [U_{nl}(r)\hat{\mathbf{r}} + V_{nl}(r)\nabla_1]Y_l^m(\theta, \phi) - W_{nl}(r)\,\hat{\mathbf{r}} \times \nabla_1 Y_l^m(\theta, \phi), \tag{4}$$

where $Y_l^m(\theta, \phi)$ are spherical harmonics, and ∇_1 is the gradient operator on the unit sphere. Spheroidal eigenfunctions have $W = 0$, while toroidal eigenfunctions have $U = V = 0$.

An example spectrum obtained for a large earthquake is presented in Figure 1. It shows well resolved spectral peaks, despite the short duration of the observation (two days only). The spheroidal eigenfrequencies are the same for the longitudinal (R) and vertical (Z) components of motion, while the toroidal eigenfrequencies are given by the transverse (T) component of the displacement. The determination of a large collection of eigenfrequencies enables us to determine more accurate spherically symmetric earth structure models, at first with an isotropic parameterization (characterized by density, P-wave and S-wave velocities) and more recently for radially anisotropic parameterization with five elastic parameters in addition to density (PREM: Preliminary Reference Earth Model) [10]. These models confirm the solid nature of the inner core and provide an estimate of S-wave velocity there.

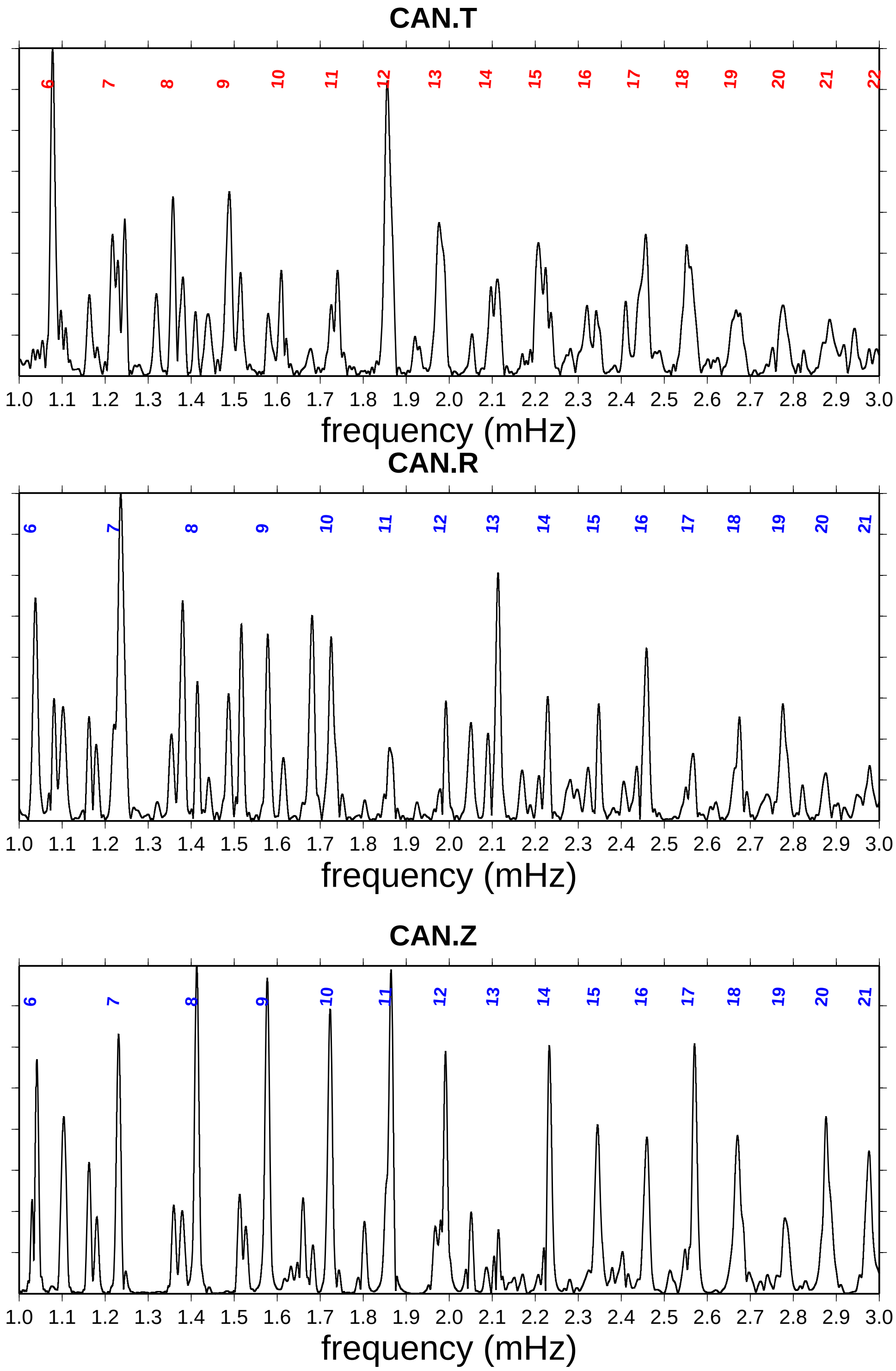

Figure 1. Spcectra of the transverse (T), longitudinal (R), and vertical components of motion for the 48 hr seismograms recorded at the GEOSCOPE station CAN (Canberra, Australia) during the Sumatra-Andaman earthquake (26 Dec 2004, Mw=9.3). The seismograms were corrected for the instrumental response. They display the two families of normal modes, spheroidal ${}_nS_l$ and toroidal ${}_nT_l$ modes. The angular degrees, l, of the fundamental modes (red for toroidal modes and blue for spheroidal modes) are drawn at their respective frequencies.

A very important point of normal mode theory is that the basis of eigenfunctions is complete. This implies that any displacement at the surface of the Earth can be expressed as a linear combination of the eigenfunctions:

$$\mathbf{u}(\mathbf{r},t) = \mathrm{Re}\sum_k a_k \mathbf{u}_k(\mathbf{r}) e^{i\omega_k t} \tag{5}$$

These eigenfunctions can be used to calculate the three-component *synthetic* displacement at any point $\mathbf{r}$ and time t due to an applied body force density $\mathbf{f}$ representing, for example, an earthquake. A point force $\mathbf{f}$ at position $\mathbf{r}_S$ is commonly described by its associated moment tensor $\mathbf{M}$ such that

$$\mathbf{f}(\mathbf{r},t) = -\mathbf{M}\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_S)\,\Theta(t), \tag{6}$$

where Θ is the Heaviside step function. The solution of the equation (1) is given by [9; 11]:

$$\mathbf{u}(\mathbf{r},t>0) = \sum_k \omega_k^{-2}\,\mathbf{M} : \epsilon_k(\mathbf{r}_S)\;\mathbf{u}_k(\mathbf{r})\;(1-\cos\omega_k t)\;e^{-\gamma_k t}, \tag{7}$$

where ϵ_k is the strain tensor associated with the displacement $\mathbf{u}_k$, with components $\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. The double dot product means contraction over two adjacent indices, $\mathbf{M} : \epsilon = M_{ij}\epsilon_{ij}$. The imaginary part, γ_k, of the eigenfrequency is non-zero in the presence of attenuation and the exponential decay serves as a correction. This expression is used for calculating synthetic seismograms. It is also used to invert for the seismic moment tensor $\mathbf{M}$ [6].

Significant discrepancies are seen between the observed and synthetic eigenfrequencies calculated for a spherically symmetric earth model. This translates into time shifts between the observed and calculated seismograms. In order to improve on the SNREI model, it is necessary to take into account the rotation of the Earth and to remove the spherically symmetric assumption by modeling the lateral heterogeneities between the source and the receiver. The measurements of the perturbations in eigenfrequencies (or time shifts) form the basis of modern tomographic methods, which map the 3D isotropic, anisotropic, anelastic structures of the Earth from the surface down to its center. Both lateral heterogeneities and rotation remove the degeneracy of modes and split the eigenfrequencies. The rotational effect is the most important at very long period, as we shall now see.

3. Rotational splitting: the Great Sumatra-Andaman earthquake

The calculation of synthetic seismograms in an aspherical Earth usually relies on first-order perturbation theory. We only present the effect of rotation on the normal modes, but the effect of heterogeneities has also been investigated (e.g. [5]). The effect of the rotation of the Earth on the normal mode eigenfrequencies and eigenfunctions is relatively simple and some analytical results have been described elsewhere [12]. The effect of rotation is quite similar to the Zeeman effect (splitting of degenerate energy levels of a hydrogen atom by a magnetic field). In a rotating frame with constant angular velocity $\mathbf{\Omega}$, and neglecting terms of order Ω^2, the wave equation becomes

$$\rho_0(\partial_t^2 + 2\mathbf{\Omega}\times\partial_t + H_0)\mathbf{u} = \mathbf{f}. \tag{8}$$

The way in which the Coriolis force affects a multiplet is approximately independent of the other multiplets, provided that this multiplet is isolated in the spectrum. This condition is not necessarily fulfilled for all modes but it is a good starting assumption. This form of theory is known as degenerate splitting theory. For an isolated multiplet, the result of first-order degenerate splitting theory is

$$\omega_{nlm} = \omega_{nl}^0 + \frac{m}{l(l+1)}\Omega\,\beta_{nl}, \tag{9}$$

where ω^0_{nl} is the frequency in the non-rotating model and β_{nl} is a factor which is different according to the type of mode (toroidal or spheroidal). For the gravest modes (frequencies smaller than 1 mHz), the effect of rotation is larger than the effect of lateral heterogeneities.

The occurence of the great Sumatra-Andaman earthquake of magnitude 9.3 represents an unprecedented opportunity to investigate the splitting of the modes that are not otherwise excited by smaller earthquakes. A systematic study of the modes from the Sumatra earthquake was carried out by Roult et al. [17]. Here we present observations for two particular multiplets. Figure 2 shows the multiplet ${}_0S_2$ and Figure 3 the multiplet ${}_3S_2$. The first multiplet, ${}_0S_2$, is primarily sensitive to the mantle down to 2900 km depth, whereas the second multiplet is very sensitive to the inner core. The prediction of the frequencies of the individual singlets split by rotation using first-order perturbation theory is given by the vertical ticks in each frame. This theory explains very well the singlets of ${}_0S_2$, but not the singlets of ${}_3S_2$. In the case of ${}_3S_2$, the observed splitting is larger than the splitting predicted by rotation (and ellipticity). Several explanations can be proposed to explain this anomalous splitting. The most likely explanation is the existence of a strong anisotropy in the inner core [5], although the origin of such anisotropy is still controversial [18].

Another application of rotational splitting has been the measurement of the differential rotation of the inner core. Several body wave studies, e.g. [19], had claimed that the inner core had a different rotation rate than the mantle by about 3° per year. However by comparing normal mode measurements over the last twenty years, Laske and Masters [13] found that this rotation difference is less than $0.13^\circ \pm 0.11^\circ$ yr^{-1}, which is consistent with other seismic data [20] and with the idea that the inner core is gravitationally locked to the mantle. This issue is not completely solved yet.

4. Conclusions

Normal mode theory and observations of the eigenfrequencies and amplitudes of the modes have played a key role in seismology in the last decades. Normal modes enable to calculate accurate synthetic seismograms and to derive the centroid moment tensor of earthquakes. Over the last five years, purely numerical methods have developed very rapidly thanks to the increasing power of supercomputers and, in the near future, will make normal mode calculations less attractive. The occurrence of the giant Sumatra-Andaman earthquake of 2004, however, revived interest in the normal modes. So did the observation of the so-called seismic "hum" [15; 21], i.e. the observation that normal modes are continuously excited even in the absense of earthquakes. The favoured explanation for the seismic hum is the interaction between the ocean and the solid Earth [16]. While seismic noise in the microseismic band (1–20 s periods) is now used for doing tomography of the crust and upper mantle [1], it is expected that the seismic hum will be used for the tomography of the deep Earth.

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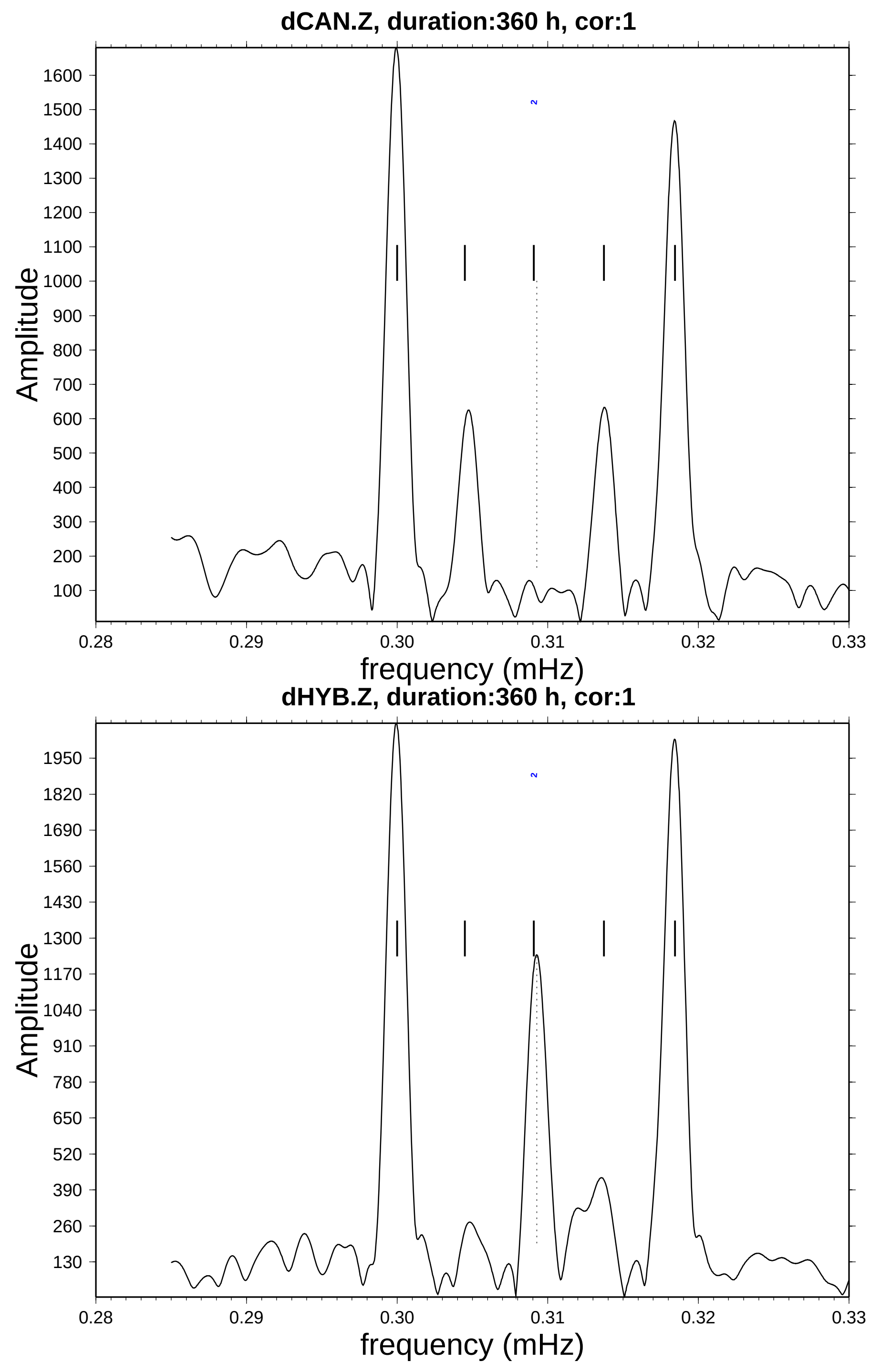

Figure 2. Split modes ${}_0S_2$ at two GEOSCOPE stations, CAN (Canberra, Australia) and HYB (Hyderabad, India). The vertical ticks correspond to the prediction of the splitting due to rotation according to first-order perturbation theory. After Ref. [17].

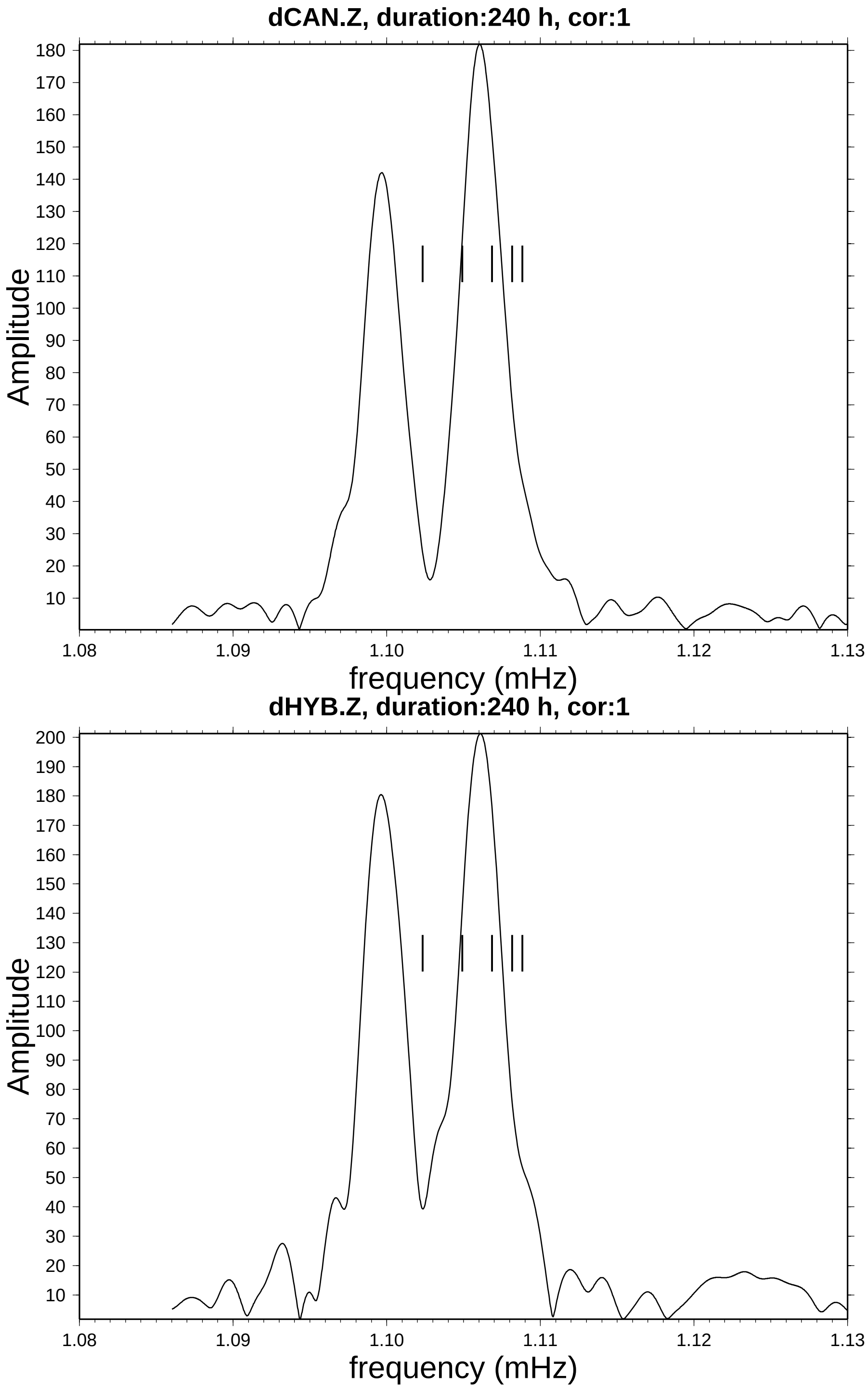

Figure 3. Same as Figure 2 but for the split spheroidal modes ${}_3S_2$ at the GEOSCOPE stations CAN (top) and HYB (bottom). The observations and the predictions (vertical ticks, first-order perturbation theory) do not agree. After Ref. [17].

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