

Image and Reciprocal Image of a Measure. Compatibility Theorem.

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Abstract

It is proposed that to the usual probability theory, three definitions and a new theorem are added, the resulting theory allows one to displace the central role usually given to the notion of conditional probability. When a mapping φ is defined between two measurable spaces, to each measure μ introduced on the first space, there corresponds an image $\varphi[\mu]$ on the second space, and, reciprocally, to each measure ν defined on the second space the corresponds a reciprocal image $\varphi^{-1}[\nu]$ on the first space. As the intersection \cap of two measures is easy to introduce, a relation like $\varphi[\mu \cap \varphi^{-1}[\nu]] = \varphi[\mu] \cap \nu$ makes sense. It is, indeed, a theorem of the theory. This theorem gives mathematical consistency to inferences drawn from physical measurements.

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1 Preliminary

Assume given a measurable space¹ (Ω, \mathcal{F}) , and a measure² μ on (Ω, \mathcal{F}) that is σ -finite³. Then, $(\Omega, \mathcal{F}, \mu)$ is called a σ -finite measure space. Let ν be a second measure on (Ω, \mathcal{F}) . The following assertions —a symmetric version of the Radon-Nikodym theorem— are equivalent (Schilling, 2006).

- The measure ν is absolutely continuous⁴ with respect to μ .
- There is a μ -almost everywhere unique function from Ω into $[0, \infty)$, denoted $d\nu/d\mu$, such that

$$\nu[F] = \int_F \frac{d\nu}{d\mu} d\mu \quad \text{for every } F \in \mathcal{F} \quad . \quad (1)$$

The function $d\nu/d\mu$ is called the Radon-Nikodym *density* associated with ν by μ , or the Radon-Nikodym *derivative* of ν with respect to μ .

If a σ -finite measure μ is such that $\mu[\Omega] = 1$, one says that μ is a *probability measure*, and the measure $\mu[F]$ of some set $F \in \mathcal{F}$ is then called the probability of the set⁵ F .

2 Definitions and properties

2.1 Intersection of measures

Given a σ -finite measure space $(\Omega, \mathcal{F}, \mu)$, consider two σ -finite measures ν_1 and ν_2 , at least one of them —say ν_1 — being absolutely continuous with respect to the base measure μ .

¹As usual, here Ω denotes a set and \mathcal{F} is a collection of subsets of Ω that is a σ -field (i.e., \mathcal{F} is nonempty and it is closed under complementation and countable unions of its members).

²A (positive) measure (measures are implicitly assumed to be positive) is a function $\mu : \mathcal{F} \mapsto [0, \infty]$ satisfying two properties: (i) the measure of the empty set is zero, and (ii) the measure of the union of a countable sequence of pairwise disjoint sets in \mathcal{F} equals the sum of the measures of each set.

³A (positive) measure μ defined on a σ -algebra \mathcal{F} of subsets of a set Ω is called σ -finite if Ω is the countable union of measurable sets of finite μ -measure. A set in a measure space has σ -finite measure if it is a countable union of sets with finite measure. A measure μ is called finite if $\mu[\Omega]$ is a finite real number (rather than ∞). A σ -finite measure may not be finite (the Lebesgue measure on the real line is σ -finite, but not finite). An example of a measure on the real line that is not σ -finite is the *counting measure* (the counting measure of a set of real numbers is the number of elements in the set): every set with finite measure contains only finitely many real numbers, and it would take uncountably many such sets to cover the entire real line. The Radon-Nikodym theorem does not apply to the counting measure and no density can be associated to it. It is to avoid such “pathological” measures that the σ -finite hypothesis is introduced.

⁴The measure ν is said to be *absolutely continuous* with respect to the measure μ if $\mu[F] = 0 \Rightarrow \nu[F] = 0$. One writes $\nu \ll \mu$.

⁵When dealing with probability measures only, sets are often called *events*.

Definition: Given some finite constant n , the intersection of the two measures ν_1 and ν_2 , is the measure denoted $\nu_1 \cap \nu_2$ and defined as

$$\boxed{(\nu_1 \cap \nu_2)[F] = \frac{1}{n} \int_F \frac{d\nu_1}{d\mu} d\nu_2 \quad \text{for every } F \in \mathcal{F} \quad .} \quad (2)$$

It is obvious that this defines a measure and that —by virtue of the Radon-Nikodym theorem— it is absolutely continuous w.r.t. ν_2 . The operation \cap depends on the base measure μ , so, when necessary, a more explicit notation, like $\bar{\cap}$, can be used.

Remark: When dealing with arbitrary measures, one may well take $n = 1$, while when dealing with probability measures, it may be more convenient to take $n = \int_{\Omega} (d\nu_1/d\mu) d\nu_2$ as, then, $(\nu_1 \cap \nu_2)[\Omega] = 1$, this implying that the intersection of two probability measures is a probability measure (but, then, the intersection would only be defined if $0 < n < \infty$).

Should the measure ν_2 also be absolutely continuous with respect to μ , equation (2) could be written $(\nu_1 \cap \nu_2)[F] = \frac{1}{n} \int_F \frac{d\nu_1}{d\mu} \frac{d\nu_2}{d\mu} d\mu$, the measure $\nu_1 \cap \nu_2$ would also be absolutely continuous with respect to μ , and its Radon-Nikodym density would be

$$\frac{d(\nu_1 \cap \nu_2)}{d\mu} = \frac{1}{n} \frac{d\nu_1}{d\mu} \frac{d\nu_2}{d\mu} \quad . \quad (3)$$

Comment: The term *intersection* is justified in section 3.2, when the intersection of (measurable) sets is found as a special instance of the intersection of measures. Another special instance of the intersection of (probability) measures corresponds to the notion of conditional probability (see section 3.3).

2.2 Reciprocal image of a measure

Let (X, \mathcal{E}) and (Y, \mathcal{F}) be two measurable spaces, and $\varphi : X \mapsto Y$ a measurable⁶ mapping. Two measures μ and ν are introduced (to be considered as base measures) such that (X, \mathcal{E}, μ) and (Y, \mathcal{F}, ν) are σ -finite measure spaces.

Definition: Given some finite constant n , to every measure τ on (Y, \mathcal{F}) that is absolutely continuous with respect to ν , is associated a measure on (X, \mathcal{E}) , called the reciprocal image of τ , denoted $\varphi^{-1}[\tau]$, and defined via

$$\boxed{\frac{d(\varphi^{-1}[\tau])}{d\mu} = \frac{1}{n} \left(\frac{d\tau}{d\nu} \circ \varphi \right) \quad .} \quad (4)$$

Then, for every $E \in \varphi^{-1}[\mathcal{F}] \subseteq \mathcal{E}$, one has⁷ $(\varphi^{-1}[\tau])[E] = \frac{1}{n} \int_E \frac{d\tau}{d\nu}(\varphi(x)) d\nu(x)$, and the Radon-Nikodym theorem ensures that $\varphi^{-1}[\tau]$ is, indeed, a measure. As this reciprocal image depends on the two base measures, the more explicit notation $\varphi^{-1}[\tau; \mu, \nu]$ can be used.

⁶ The mapping φ is measurable, if the reciprocal image of every set in \mathcal{F} is in \mathcal{E} . Non measurable mappings are generally considered pathological.

⁷ As φ is a measurable mapping, and the function $d\tau/d\nu$ is measurable (with respect to \mathcal{F}), the function $(d\tau/d\nu) \circ \varphi$ is measurable (with respect to $\varphi^{-1}[\mathcal{F}] \subseteq \mathcal{E}$). (See, e.g., Halmos, 1950).

Remark: When dealing with arbitrary measures, one may well take $n = 1$, while when dealing with probability measures, it may be more convenient to take $n = \int_X \frac{d\tau}{d\nu} \circ \varphi d\mu$, as, then, $(\varphi^{-1}[\tau])[X] = 1$, this implying that the reciprocal image of a probability measure is a probability measure (but, then, the reciprocal image would only be defined if $0 < n < \infty$).

2.3 Image of a measure

Let (X, \mathcal{E}) and (Y, \mathcal{F}) be two measurable spaces, and $\varphi : X \mapsto Y$ a measurable mapping.

Definition: To every measure π on (X, \mathcal{E}) , is associated a measure⁸ on (Y, \mathcal{F}) , denoted $\varphi[\pi]$, called the image measure:

$$\boxed{\varphi[\pi] = \pi \circ \varphi^{-1} \quad ,} \quad (5)$$

i.e., explicitly, $(\varphi[\pi])[F] = \pi[\varphi^{-1}[F]]$ for every $F \in \mathcal{F}$.

The measure $\varphi[\pi]$ needs not be⁹ absolutely continuous with respect to some base measure, so $\varphi[\pi]$ may not be representable by a bona-fide density¹⁰. This does not cause any complication in the applications we have in mind.

We shall later need the following property (Halmos, 1950, page 163): for any measurable function K and any set $F \in \mathcal{F}$,

$$\int_F K d(\pi \circ \varphi^{-1}) = \int_{\varphi^{-1}[F]} K \circ \varphi d\pi \quad , \quad (6)$$

i.e., $\int_F K(y) d(\varphi[\pi])(y) = \int_{\varphi^{-1}[F]} K(\varphi(x)) d\pi(x)$.

Comment: To have an intuitive idea of the notion “image of a measure”, consider a collection of elements x_1, x_2, x_3, \dots of X that are independent sample elements of the measure π . Then, it is easy to see that the elements $\varphi(x_1), \varphi(x_2), \varphi(x_3), \dots$ of Y are independent sample elements of the measure $\varphi[\pi]$. In fact, this property alone may suggest introducing the notion of an image of a measure.

⁸ It is not difficult to verify that $\varphi[\pi]$ is, indeed, a measure. First, the measure of the empty set is zero, because, by definition of the reciprocal mapping, $\varphi^{-1}[\emptyset] = \emptyset$, so $(\varphi[\pi])[\emptyset] = \pi[\varphi^{-1}[\emptyset]] = \pi[\emptyset] = 0$. Second, we have to check that if F_1, F_2, \dots is a countable sequence of pairwise disjoint sets in \mathcal{F} the measure of the union of all the F_i is equal to the sum of the measures of each F_i : $(\varphi[\pi])[\bigcup_i F_i] = \sum_i \tau[F_i]$. First, by definition of image of a measure, one has $(\varphi[\pi])[\bigcup_i F_i] = \pi[\varphi^{-1}[\bigcup_i F_i]]$ and, as the reciprocal image of a union is the union of the reciprocal images, $(\varphi[\pi])[\bigcup_i F_i] = \pi[\bigcup_i \varphi^{-1}[F_i]]$. But π is a measure, and the reciprocal image of disjoint sets is disjoint, so $(\varphi[\pi])[\bigcup_i F_i] = \sum_i \pi[\varphi^{-1}[F_i]]$. Finally, using again the definition of image of a measure, this leads to desired property. We have thus checked that the image of a measure is a measure.

⁹ As an example, this happens when $X = \mathbb{R}^p$ and $Y = \mathbb{R}^q$ with $p < q$, and a continuous mapping φ (with the standard Lebesgue measures assumed), because then $\varphi[X]$ is a p -dimensional submanifold of \mathbb{R}^q .

¹⁰ When $\varphi[\pi]$ is representable by a density, it is, in general, easy to find an expression of it, but the (elementary) methods to be used are quite different in every situation (see examples in sections 4.2 and 4.3), and a general expression for the density is not available.

2.4 Compatibility property

Let (X, \mathcal{E}, μ) and (Y, \mathcal{F}, ν) be two σ -finite measure spaces, and $\varphi : X \mapsto Y$ be a measurable mapping. Let π be a measure over (X, \mathcal{E}) that is σ -finite, and τ a measure over (Y, \mathcal{F}) , that is absolutely continuous with respect to the base measure ν .

Theorem: *One always has*

$$\boxed{\varphi[\pi \cap \pi'] = \tau' \cap \tau \quad \text{where} \quad \begin{cases} \pi' = \varphi^{-1}[\tau] \\ \tau' = \varphi[\pi] \end{cases}} \quad (7)$$

Note that while the measure τ is assumed to be absolutely continuous with respect to the base measure ν , the measures $\varphi[\pi]$ and $\varphi[\pi \cap \pi']$ may be singular.

To demonstrate the identity in equation (7) means to verify that for any set $F \in \mathcal{F}$, one has $(\varphi[\pi \cap \pi'])[F] = (\tau' \cap \tau)[F]$. This is done by writing the following sequence of identities (that successively use equations (5), (2), (4), (6), and (2) again):

$$\begin{aligned} (\varphi[\pi \cap \varphi^{-1}[\tau; \mu, \nu]])(F) &= (\pi \cap \varphi^{-1}[\tau; \mu, \nu])(\varphi^{-1}[F]) \\ &= \frac{1}{n} \int_{\varphi^{-1}[F]} \frac{d(\varphi^{-1}[\tau; \mu, \nu])}{d\mu}(x) d\pi(x) \\ &= \frac{1}{n'} \int_{\varphi^{-1}[F]} \frac{d\tau}{d\nu}(\varphi(x)) d\pi(x) \\ &= \frac{1}{n'} \int_F \frac{d\tau}{d\nu}(y) d(\varphi[\pi])(y) \\ &= \frac{1}{n''} (\varphi[\pi] \cap \tau)(F) \quad , \end{aligned} \quad (8)$$

so the property holds¹¹.

3 Measures and sets

3.1 Measure-sets

The definitions and properties above have a direct relation with definitions and properties in set theory, and, in some sense, they generalize them. To see this, let us start by introducing the notion of measure-set.

Let $(\Omega, \mathcal{F}, \mu)$ be a σ -finite measure space. To every set $A \in \mathcal{F}$ we shall associate a measure, denoted μ_A , and defined via the condition

$$\mu_A[F] = \frac{1}{n_A} \mu[A \cap F] \quad \text{for every } F \in \mathcal{F} \quad , \quad (9)$$

¹¹ The constant n'' equals one, because for general measures the two constants in equations (2) and (4) should be taken equal to one, while for probability measures, there is an automatic renormalization.

where n_A is a suitable chosen constant (that may depend on A but not on F). As suggested above, one may well take

$$n_A = 1 \quad \text{for arbitrary measures} \quad , \quad (10)$$

or

$$n_A = \mu[A] \quad \text{for probability measures} \quad , \quad (11)$$

because, then, $\mu_A[\Omega] = 1$ (of course, this assumes $\mu[A] \neq 0$.) Such a measure shall be called a *measure-set*, so we can talk about the measure-set μ_A associated with a set A by a measure μ . The μ -density associated with a measure-set μ_A is clearly¹² $\frac{1}{n_A} \chi_A$, i.e., proportional to the characteristic function of the set A .

Of course, there may be subsets of Ω that are not in \mathcal{F} , but, as far as one is only interested in the sets in \mathcal{F} , one can consider that any measure absolutely continuous w.r.t. μ is something like a generalized set: while a measure-set can be identified to a set (its density taking only two possible values, 0 or $1/n_A$), an arbitrary measure (with a density taking any nonnegative value) is a kind of generalized object, that contains measure-sets and, therefore, sets as special cases.

The names given to the three notions introduced above —intersection of measures, and image and reciprocal image of a measure— are justified because (as we are about to see) when applied to measure-sets they do correspond to the intersection, the image and the reciprocal image of sets.

3.2 Measures versus sets: intersection

If A and B are two sets in \mathcal{F} and μ_A and μ_B are the two associated measure-sets, one has¹³

$$\mu_A \cap \mu_B = k \mu_{A \cap B} \quad (12)$$

with the constant¹⁴ $k = \frac{n_{A \cap B}}{n_A n_B}$. So —in the special case where the measures are measure-sets— the definition of intersection of measures is consistent with the definition of intersection of sets.

3.3 Intersection of measures and conditional probability

Letting A be a fixed set of \mathcal{F} , let us now consider the intersection of an arbitrary measure ν and the measure-set μ_A , i.e., the measure $\nu \cap \mu_A$. One has¹⁵

$$(\nu \cap \mu_A)[F] = \frac{1}{n} \nu[F \cap A] \quad \text{for every } F \in \mathcal{F} \quad , \quad (13)$$

¹² As, for every $F \in \mathcal{F}$, $\frac{1}{n_A} \int_F \chi_A d\mu = \frac{1}{n_A} \int_{A \cap F} d\mu = \frac{1}{n_A} \mu(A \cap F) = \mu_A[F]$.

¹³ For any $F \in \mathcal{F}$, $(\mu_A \cap \mu_B)[F] = \frac{1}{n_A n_B} \int_F \chi_A \chi_B d\mu = \frac{1}{n_A n_B} \int_{F \cap A \cap B} d\mu = \frac{n_{A \cap B}}{n_A n_B} \int_F \chi_{A \cap B} d\mu = \frac{n_{A \cap B}}{n_A n_B} \mu_{A \cap B}[F]$.

¹⁴ For general measures, $k = 1$, while for probability measures, $k = \frac{\mu[A \cap B]}{\mu[A] \mu[B]}$.

¹⁵ For $(\nu \cap \mu_A)[F] \propto \int_F \chi_A d\nu = \int_{F \cap A} d\nu = \nu[F \cap A]$

where n is a constant. This is particularly interesting when dealing with probability measures, because, then, $n = \nu[A]$, and one has

$$(\nu \cap \mu_A)[F] = \frac{\nu[F \cap A]}{\nu[A]} \quad \text{for every } F \in \mathcal{F} \quad . \quad (14)$$

One immediately recognizes there the expression of Kolmogorov's conditional probability, usually denoted $\nu[F \cap A]/\nu[A] = \nu[F | A]$. Using this notation,

$$(\nu \cap \mu_A)[F] = \nu[F | A] \quad \text{for every } F \in \mathcal{F} \quad . \quad (15)$$

So we have the following

Property: *For every given probability measure ν , Kolmogorov's conditional probability, given any set A , is identical to the intersection of ν by the measure-set μ_A .*

So we see that the notion of conditional probability is a special case of the notion of intersection of measures: when evaluating the intersection of an arbitrary measure by a measure set, we have the conditional probability, but we can evaluate the intersection of two general measures. I claim that there are problems that are naturally formulated in terms of the intersection of two measures (see sections 4.1 and 4.5 for examples). As this notion has not been available so far, some hand-waving has been necessary to make this kind of problems fit into the available mathematical structure. This, plus the fact that general mappings between arbitrary sets (as opposed to linear mappings between linear spaces) can be taken as root elements, is what has motivated the building of the present theory.

3.4 Measures versus sets: reciprocal images

When considering a mapping φ from a set X into a set Y to every set $B \subseteq Y$ there is associated a subset of X , denoted $\varphi^{-1}[B]$ and named the reciprocal image¹⁶ of the set B . But the reciprocal image of a set can also be defined in terms of the characteristic functions of the sets: letting ξ_A the characteristic function of a set $A \subseteq X$ and χ_B that of a set $B \subseteq Y$, one clearly has

$$\xi_{\varphi^{-1}[B]} = \chi_B \circ \varphi \quad , \quad (16)$$

a relation that is identical (with $n = 1$) to the relation (4) expressing the reciprocal image of a measure in terms of Radon-Nikodym densities. So, as it already happened with the intersection of measures, the notion of reciprocal image of a measure is consistent with the definition of reciprocal image of a set: the set associated to the reciprocal image of the measure-set that is associated to a set B is the reciprocal image of the set B :

$$\varphi^{-1}[\nu_B] = \mu_{\varphi^{-1}[B]} \quad . \quad (17)$$

In this sense, again, the notion of reciprocal image of a set is “contained” in the notion of reciprocal image of a measure.

¹⁶ The set $\varphi^{-1}[B]$ is made of all the elements of X whose image is in B .

3.5 Measures versus sets: images

The relation between the notion of image of a set (in set theory) and the notion of image of a measure is subtle. In this short note, let us just mention that the support¹⁷ of the image of a measure $d(\varphi[\pi])/d\nu$ is the image (in the sense of set theory) of the support of the original measure $d\pi/d\mu$.

3.6 Measures versus sets: compatibility property

In set theory, for arbitrary sets A_1 and A_2 and for an arbitrary mapping φ , one has

$$\varphi[A_1 \cap A_2] \subseteq \varphi[A_1] \cap \varphi[A_2] \quad , \quad (18)$$

a relation that is well-known but not very useful here. A more useful relation (for making inferences involving sets and mappings) is that, for arbitrary sets A and B , and an arbitrary mapping φ , one has

$$\varphi[A \cap \varphi^{-1}[B]] = \varphi[A] \cap B \quad . \quad (19)$$

For reasons that shall become clear in the applications (see section 4.5), it is interesting to extend this identity into probability theory (or, more generally, inside measure theory). But, of course, our compatibility property (equation (7))

$$\varphi[\pi \cap \varphi^{-1}[\tau]] = \varphi[\pi] \cap \tau \quad , \quad (20)$$

is identical to relation (19), excepted that it concerns measures instead of sets. So, in some sense, we have generalized the set relation. In any case, when the relation $\varphi[\pi \cap \varphi^{-1}[\tau]] = \varphi[\pi] \cap \tau$ is applied to measure sets, it becomes relation (19).

4 Applications

4.1 Intersection of probability measures

Let Ω represent the surface of the sphere of unit radius, and \mathcal{B} the usual Borel collection of subsets¹⁸. Consider, on the measurable space (S, \mathcal{B}) , the ordinary surface measure: for any set $B \in \mathcal{B}$ of points on the sphere, $S[B]$ is the surface of B . Two probability measures P_1 and P_2 are then considered, and two points \mathcal{P}_1 and \mathcal{P}_2 are randomly created¹⁹ on the surface of the sphere, that are random point samples of the respective probability measures P_1 and P_2 . If $\mathcal{P}_1 \neq \mathcal{P}_2$ the two points are discarded, and two new points are generated. And so on until the two points happen to be identical, $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}$.

Question: of which probability measure is \mathcal{P} a random point sample?

¹⁷ The support of a function is the set of points where the function is not zero.

¹⁸The Borel sigma-field is defined as the smallest sigma-field containing all the open subsets.

¹⁹The notion of random point is not introduced here; it is assumed that reader knows the basic notion of sampling from a probability measure.

Answer: point \mathcal{P} is a random point sample of the probability measure

$$P = P_1 \cap P_2 \quad . \quad (21)$$

Proof: The probability that the two points \mathcal{P}_1 and \mathcal{P}_2 happen to be identical is zero, so the question makes no immediate sense, and needs to be slightly reformulated. If the sphere is assumed to be tiled with a finite collection of (spherical) tiles of identical surface ΔS , then it can happen that the two points \mathcal{P}_1 and \mathcal{P}_2 are in the same tile. The finite probability that this happens in a given tile can then be evaluated (it is the renormalized product of the two probabilities assigned to the tile by each of the two probability measures P_1 and P_2), and it is when taking the limit $\Delta S \rightarrow 0$ that one gets the result.

Introducing the three probability densities f_1 , f_2 , and $f_1 \cap f_2$ associated with the three probability measures P_1 , P_2 , and $P_1 \cap P_2$ via

$$P_1[B] = \int_B f_1 dS \ ; \ P_2[B] = \int_B f_2 dS \ ; \ (P_1 \cap P_2)[B] = \int_B (f_1 \cap f_2) dS \ , \quad (22)$$

gives here, using equation (3),

$$(f_1 \cap f_2) = \frac{f_1 f_2}{\int_{\Omega} f_1 f_2 dS} \quad . \quad (23)$$

To pass from this purely mathematical exercise to a problem involving real-life measurements, assume that two totally “disentangled”²⁰ measurements of the position of a floating object on the ocean provide the information described (following ISO’s recommendations [ISO, 1993]) by two probability densities f_1 and f_2 . How should they be “combined” to represent the total available information? The detailed justifications of this is outside the scope of this short note, but I suggest here that experimental uncertainties are defined in such a way (ISO’s way) that the answer to the question is precisely that in equation (23).

4.2 Mapping between discrete sets

Let X and Y be discrete spaces, \mathcal{E} and \mathcal{F} the respective collections of their subsets, and φ a mapping from X into Y . Consider that a probability measure π on (X, \mathcal{E}) , is sampled, this providing elements x_1, x_2, \dots of X , and therefore, via the mapping φ , the image elements $y_1 = \varphi(x_1), y_2 = \varphi(x_2), \dots$ of Y . Of which probability measure τ on (Y, \mathcal{F}) are the elements y_1, y_2, \dots sample points?

The answer is $\tau = \varphi[\pi]$, as this clearly corresponds to the very definition of image of a measure (equation (5)):

$$\tau[F] = \pi[\varphi^{-1}[F]] \quad \text{for every } F \in \mathcal{F} \quad . \quad (24)$$

²⁰ I am trying here to avoid the use of the term *independent* that has a related —but different— connotation in probability theory.

To transform this result into an explicit expression, we can introduce two base measures μ and ν on (X, \mathcal{E}) and (Y, \mathcal{F}) respectively, for instance, the respective counting measures²¹. The density f associated with the measure π consists then in the (discrete) collection of numbers $f_i = f(x_i)$ such that, for every set $E \subseteq X$, $\pi[E] = \int_E f d\mu = \sum_{x_i \in E} f(x_i)$, while the density g (that we may denote $g = \varphi[f]$) associated to the measure $\tau = \varphi[\pi]$ consists in the (discrete) collection of numbers $g_\alpha = g(y_\alpha)$ such that for every set $F \subseteq Y$ $\tau[F] = \int_F g d\nu = \sum_{y_\alpha \in F} g(y_\alpha)$. Some easy computations then provide the solution:

$$g(y_\alpha) = \sum_{x_i \in \varphi^{-1}\{y_\alpha\}} f(x_i) \quad \text{for every } y_\alpha \in Y \quad . \quad (25)$$

4.3 Propagation of uncertainties in physical measurements

Physical quantities are often defined in terms of other physical quantities. For instance, the electric resistance R of a wire is defined as the ratio of the voltage V applied to the wire and the current intensity I flowing in the wire. Then, a typical measure of R involves, in fact, the measure of the two quantities V and I and the computation of the ratio $R = V/I$.

So, more generally, when one wants to perform a physical measurement of the value of some physical quantity, say y , most of the time, one resorts to measuring in fact some other quantities, say $\{x^1, x^2, \dots, x^p\}$, and then one computes the value of y via its definition

$$y = \varphi(x^1, x^2, \dots, x^p) \quad . \quad (26)$$

One very basic problem in metrology is that of “propagating” the uncertainties appearing in the measurements of the quantities x^i into the uncertainty on the quantity y . Good metrology practice corresponds (ISO, 1993) to representing the uncertainties on a measurement by a probability density (as opposed to simple “uncertainty bar”). Therefore, one faces the following problem:

Question: One has some probability measure π defined on the quantities $\{x^1, x^2, \dots, x^p\}$, and one *defines* the quantity y via the mapping in equation (26). What probability measure τ does this imply on the quantity y ?

Short answer: The probability measure τ is the image of the probability measure π , i.e., according to our general definition of image of a measure: $\tau = \varphi[\pi]$.

But let us state the problem using a more general terminology.

Preliminaires: Some of the quantities $x = \{x^1, x^2, \dots, x^p\}$ may be discrete, while others may be real quantities, each taking values inside some interval (open or closed). Let X be the set (part discrete, part continuous) whose elements

²¹ The counting measure of a set is the number of elements in the set.

correspond to all the possible values of the quantities x . Introducing an appropriate σ -field of subsets of X is, generally, quite easy²², so one immediately faces a measurable space (X, \mathcal{E}) . We can consider, for more generality, that the quantity y also is “multidimensional”: $y = \{y^1, y^2, \dots, y^q\}$. A measurable space (Y, \mathcal{F}) is introduced as above. Unless that mapping $x \mapsto y = \varphi(x)$ is pathological²³, it will be measurable (with respect to the two σ -fields \mathcal{E} and \mathcal{F}). The (uncertain) result of the measurement of the quantities x is represented by a σ -finite²⁴ measure π on (X, \mathcal{E}) .

Question: How do the uncertainties encapsulated by the measure π “propagate” into uncertainties on the space (Y, \mathcal{F}) , i.e., which is the measure, say τ , implied on (Y, \mathcal{F}) by the measure π and the mapping φ ?

Answer: The notion of “propagation of uncertainties” can be made precise by imposing that the probability $\tau[B]$ of any subset $B \in \mathcal{F}$ must equal the probability of the pre-image (or reciprocal image) of the subset:

$$\tau[B] = \pi[\varphi^{-1}[B]] \quad \text{for any } B \in \mathcal{B} \quad . \quad (27)$$

But this is exactly our definition of image of a measure, so the answer is

$$\tau = \varphi[\pi] \quad . \quad (28)$$

Example: The measurement of an electric resistance R involves the measurement of the two quantities V and I and the use of the definition $R = V/I$. If the result of the measurement of V and I (and the associated uncertainties) is that represented by the (lognormal) probability density

$$f(V, I) = \frac{1}{2\pi\sigma_V\sigma_I} \frac{1}{VI} \exp\left(-\frac{\log^2(V/V_0)}{2\sigma_V^2} - \frac{\log^2(I/I_0)}{2\sigma_I^2}\right) \quad , \quad (29)$$

then, the notion of image of a measure produces, for the electric resistance R , the (lognormal) probability density

$$g(R) = \frac{1}{\sqrt{2\pi}\sigma_R} \frac{1}{R} \exp\left(-\frac{\log^2(R/R_0)}{2\sigma_R^2}\right) \quad , \quad (30)$$

where

$$R_0 = V_0/I_0 \quad \text{and} \quad \sigma_R = \sqrt{\sigma_V^2 + \sigma_I^2} \quad . \quad (31)$$

Let us see some details of that. The space X is $(0, \infty) \times (0, \infty) \subset \mathfrak{R}^2$, that we endow with two coordinates $\{V, I\}$ (having the physical interpretation of an

²² A Cartesian product of some Borel fields—for the real variables—times the collections of all the possible subsets—for the discrete variables—.

²³ Physicists need to try hard before being able to introduce mappings that are not measurable with respect to the obvious topologies.

²⁴ Physicists will typically represent their measurement uncertainties by introducing probability densities and discrete probabilities, to be interpreted as the Radon-Nikodym derivatives of the measure π , so π will be σ -finite by construction.

electric voltage and an electric intensity). The space Y is $(0, \infty) \subset \Re$ that we endow with a coordinate R (having the physical interpretation of an electric resistance). The mapping φ is (definition of electric resistance) $R = V/I$. The usual Borel σ -fields of X and of Y (say \mathcal{B}_2 and \mathcal{B}_1) are introduced, and the usual Lebesgue measures are considered as base measures. To arrive at the density $g(R)$ one can here introduce the “slack” variable $P = VI$, this allowing to consider the “change of variables” $\{V, I\} \mapsto \{R, P\}$. One then easily evaluates the density $g_0(R, P)$ (using the Jacobian of the transformation), and, from it, $g(R) = \int_0^\infty g_0(R, P) dP$. It can be shown that the final result for $g(R)$ is independent on the particular choice of slack variable.

4.4 Interpretation of observations (1: using sets)

In the physical sciences, some problems of interpretation of observations can be idealized as follows. There are two sets X and Y , a mapping φ from X into Y , and

(i) we are interested in identifying a particular element $x \in X$, and we have the “a priori information” that it belongs to a subset $X_{\text{prior}} \subseteq X$:

$$x \in X_{\text{prior}} \quad , \quad (32)$$

(ii) we have “observed” that some element $y \in Y$ belongs to a subset $Y_{\text{obs}} \subseteq Y$:

$$y \in Y_{\text{obs}} \quad , \quad (33)$$

and (iii) we know that y is related to x via the mapping φ :

$$y = \varphi(x) \quad . \quad (34)$$

These three pieces of information, when put together (see the left of figure 1), allow one to infer (using standard set theory reasoning):

(i) that the element x belongs, in fact, to a set X_{post} that is smaller or equal to the original set X_{prior} ,

$$x \in X_{\text{post}} = X_{\text{prior}} \cap \varphi^{-1}[Y_{\text{obs}}] \subseteq X_{\text{prior}} \quad , \quad (35)$$

(ii) while the element y belongs, in fact, to a set Y_{post} that is smaller or equal to the original set Y_{obs} ,

$$y \in Y_{\text{post}} = \varphi[X_{\text{post}}] \cap Y_{\text{obs}} \subseteq Y_{\text{obs}} \quad . \quad (36)$$

These two results are obvious. Perhaps less obvious is the relation

$$\boxed{Y_{\text{post}} = \varphi[X_{\text{post}}]} \quad , \quad (37)$$

that follows directly from the universal set property $\varphi[A \cap \varphi^{-1}[B]] = \varphi[A] \cap B$ (equation (19)).

Remark that we are inside the paradigm typical of a “problem of assimilation of observations” —sometimes called “inverse modeling problem”—: the mapping $x \mapsto y = \varphi(x)$ can be seen as the typical mapping between the “model parameter space” and the “observable parameter space”. In what concerns the element $x \in X$ we pass from the “a priori information” $x \in X_{\text{prior}}$ to the “a posteriori information” $x \in X_{\text{post}} \subseteq X_{\text{prior}}$. Similarly, in what concerns the element $y \in Y$ we pass from the “initial observation” $y \in Y_{\text{obs}}$ to the “refined observation” $y \in Y_{\text{post}} \subseteq Y_{\text{obs}}$.

In the next section, the same problem is reformulated, but using probability measures instead of sets.

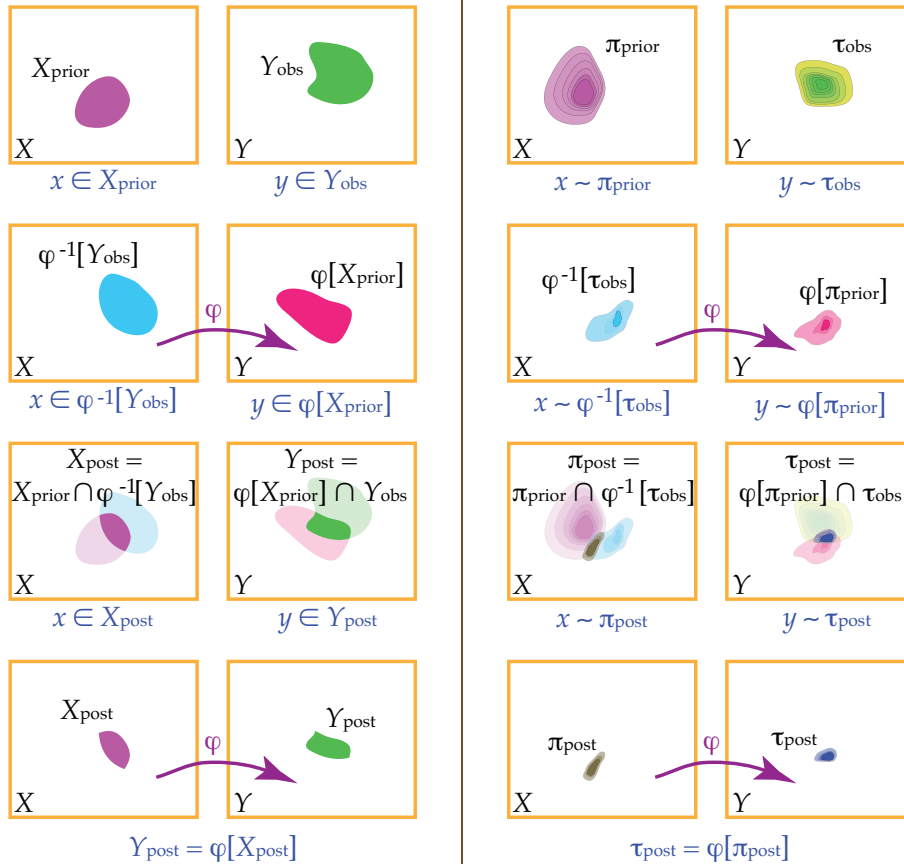


Figure 1: At the left, an inference problem that can be solved using only set theory (see text). At the right, a similar problem, but this one concerning probability measures (see text). This example at the right corresponds to many of the so-called *inverse problems* in the experimental sciences.

4.5 Interpretation of observations (2: using measures)

The problem of interpretation of observation —sometimes called the “inverse problem”— appears as follows. A physical system (e.g., a molecule, an ocean, a planet’s atmosphere, a galaxy) is under investigation. For the purposes of the investigation, the system is described using a collection of p physical quantities $x = \{x^1, x^2, \dots, x^p\}$; some of them taking only discrete values (for instance, *black* or *white*) and some others taking continuous values (for instance a temperature can take any positive real value). A set X is introduced, the elements (or “points”) of which corresponding to the quantities x taking all their possible values. In the jargon of inverse problem theory, the set X is called the *model space*²⁵. In order to gain information on the p physical quantities x , a set of q physical quantities $y = \{y^1, y^2, \dots, y^q\}$ —perhaps only quite indirectly related to the quantities x — is measured. As above, when considering all possible values for the q quantities y one is faced with a set Y , the “observable parameter space”. One then (usually implicitly) considers two collections of subsets \mathcal{E} and \mathcal{F} , of sets of X and of Y respectively, that are σ -algebras, so one has two measurable spaces (X, \mathcal{E}) and (Y, \mathcal{F}) . On each of these spaces, a base (σ -finite) measure has to be considered in order to have two σ -finite measure spaces, (X, \mathcal{E}, μ) , and (Y, \mathcal{F}, ν) . To a mathematician, the existence of the two base measures μ and ν may seem a minor hypothesis. A physicist may have to work hard to find them, as they must represent the volume measure of each space, and, as such, they must have the necessary invariances²⁶. For this reason, let us here call the two base measures μ and ν the respective *volume measures*. They matter, because the reciprocal image of a probability measure on (Y, \mathcal{F}) and the intersection of measures on (X, \mathcal{E}) and on (Y, \mathcal{F}) depend on them.

The final structure element is that a physical theory is assumed to exist, that is able —given any possible value of the model parameters x — to predict (in a Popperian sense) the observations y . This prediction consists in a mapping $\varphi : X \mapsto Y$, that must be assumed to be measurable (what, for a physicist, just means that φ is assumed to be not “pathological”). Of course, the mapping φ is not assumed to be invertible (and it may be “nonlinear”).

In a typical inverse problem one cares in introducing any available a priori information on x (that means information available before the measurements on y are carried out) as a probability measure, say π_{prior} , on (X, \mathcal{E}) (it must be “ordinary”, i.e., σ -finite, but it does not need to be absolutely continuous w.r.t. the volume measure μ). When the measurements of the quantities y are

²⁵ In fact, the set X is something more abstract: any (invertible) change of variables $x \rightleftharpoons x'$ is to be seen as a “change of coordinates” inside X , not as the definition of a new set X' . For a discussion of this kind of intrinsic view on physical quantities, see Tarantola (2006).

²⁶ For instance, one of the quantities may be the period T (of, say, a star). How to measure the volume (in fact, the length) of an interval (T_1, T_2) , say $\mu[(T_1, T_2)]$? Taking $\mu[(T_1, T_2)] = |T_2 - T_1|$ would not be consistent with, when working with the frequency $\omega = 2\pi/T$, measuring the volume as $|\omega_2 - \omega_1|$, because $|T_2 - T_1| \neq |\omega_2 - \omega_1|$. In fact, the right volume measure is $\mu[(T_1, T_2)] = |\log(T_2/T_1)|$, because $|\log(T_2/T_1)| = |\log(\omega_2/\omega_1)|$. See Tarantola (2005) for an elementary discussion of this problem, or Tarantola (2006) for a more advanced discussion.

carried out, the result is represented²⁷ as a probability measure, say τ_{obs} , on (Y, \mathcal{F}) , that must be absolutely continuous w.r.t. the volume measure ν (i.e., it has to be representable by a density). So, one has the following three elements (see the right of figure 1):

(i) a priori information on the model parameters, i.e., a probability measure on (X, \mathcal{E})

$$\pi_{\text{prior}} \quad ; \quad (38)$$

(ii) results of the measurements, i.e., a probability measure on (Y, \mathcal{F})

$$\tau_{\text{obs}} \quad ; \quad (39)$$

and (iii) the modeling mapping

$$\varphi : X \mapsto Y \quad . \quad (40)$$

It is clear that the existence of a mapping φ is going to transform the measure π_{prior} into some other measure, say π_{post} , and the measure τ_{obs} into some measure, say τ_{post} , much as it happened in section 4.4, where the prior sets were transformed into posterior sets. The problem is that here we are facing natural objects (measurements and physical laws), that are not easily amenable to axiomatization. Usual presentations of the inverse problem unconvincingly use intuitive interpretations of the notion of conditional probability and of, perhaps, Bayes' theorem. I prefer here to frankly state that I formulate the problem using only the analogy between the present problem and the set-theoretical problem in section 4.4 (although closer analogies can be elaborated²⁸). The results there (that concerned intersection of sets, and images and reciprocal images of sets) were unquestionable. As far as the present theory, defining the intersection of measures, and images and reciprocal images of measures is an acceptable generalization of (a part of) set theory (and the compatibility property suggests that it is), we can match this problem to the one in section 4.4 (see also the parallel suggested in figure 1). In any case, the formulas we are going to find for the inverse problem are basically identical (although, perhaps, a little more general) than those proposed in the usual literature²⁹.

Then (see the illustration at the right of figure 1):

(i) on the model parameter space (X, \mathcal{E}) , one passes from the prior probability measure π_{prior} to the posterior probability measure

$$\pi_{\text{post}} = \pi_{\text{prior}} \cap \varphi^{-1}[\tau_{\text{obs}}] \quad ; \quad (41)$$

²⁷ The representation of the (possibly uncertain) result of an observation as a probability measure is in compliance with ISO's (1993) recommendations.

²⁸ Assume that a random x and a random y are created according respectively to π_{prior} and τ_{obs} , and that the pair $\{x, y\}$ is accepted only if $y = \varphi(x)$ (much as we did in section 4.1). It is easy to prove (see the argument in section 4.1) that when a pair $\{x, y\}$ is accepted, x is a sample point of the measure $\pi_{\text{post}} = \pi_{\text{prior}} \cap \varphi^{-1}[\tau_{\text{obs}}]$ and y is a sample point of the measure $\tau_{\text{post}} = \varphi[\pi_{\text{prior}}] \cap \tau_{\text{obs}}$. These are exactly expressions (41) and (42).

²⁹ For a probabilistic formulations of the inverse problem, see Tarantola and Valette (1982), Menke (1989), Mosegaard and Tarantola (1995), Aster et al. (2005), or Tarantola (2005). For an alternative, statistical decision theory, see Evans and Stark (2002).

(*ii*) on the observable parameter space (Y, \mathcal{F}) , one passes from the initial probability measure τ_{obs} (representing the result of the measurements) to the probability measure

$$\tau_{\text{post}} = \varphi[\pi_{\text{prior}}] \cap \tau_{\text{obs}} \quad (42)$$

representing a refined estimation of the values of the observable parameters y . Finally, (*iii*) the compatibility property (equation (7)) nicely states that

$$\tau_{\text{post}} = \varphi[\pi_{\text{post}}] \quad . \quad (43)$$

Let us evaluate the posterior probability $\pi_{\text{post}}[E]$ of some set $E \in \mathcal{E}$. From expression (41) it follows (using, first, the definition of the intersection of measures in equation (2), then, the definition of the reciprocal image of a measure in equation (4))

$$\pi_{\text{post}}[E] = \frac{1}{n} \int_E \frac{d\tau}{d\nu}(\varphi(x)) d\pi_{\text{prior}} \quad , \quad (44)$$

where the constant $n = \int_X \frac{d\tau}{d\nu}(\varphi(x)) d\pi_{\text{prior}}$ must be different from zero (in order for π_{post} to be a probability measure). Note that the density $d\tau/d\nu$ exists because τ was assumed to be absolutely continuous w.r.t. the volume measure ν .

To evaluate the posterior probability $\tau_{\text{post}}[F]$ of a set $F \in \mathcal{F}$, one could try to start with expression (42), but this possibility is not the most practical. One can rather use the compatibility relation (43), as then (because of the relation (4) defining the image of a measure),

$$\tau_{\text{post}}[F] = \pi_{\text{post}}[\varphi^{-1}[F]] \quad . \quad (45)$$

In real-life problems, the finite probabilities $\pi_{\text{post}}[E]$ and $\tau_{\text{post}}[F]$ (i.e., the sums in equations (44)–(45) can (approximately) be evaluated using Monte Carlo methods³⁰.

As a final remark, should the prior probability measure π_{prior} be absolutely continuous w.r.t. the volume measure μ (i.e., should the density $d\pi_{\text{prior}}/d\mu$ exist), the posterior probability measure π_{post} would also have a density, whose explicit expression would follow immediately from equation (44):

$$\frac{d\pi_{\text{post}}}{d\mu} = \frac{1}{n} \frac{d\pi_{\text{prior}}}{d\mu} \frac{d\tau}{d\nu} \circ \varphi \quad , \quad (46)$$

i.e., explicitly, $\frac{d\pi_{\text{post}}}{d\mu}(x) = \frac{1}{n} \frac{d\pi_{\text{prior}}}{d\mu}(x) \frac{d\tau}{d\nu}(\varphi(x))$. In the jargon of inverse theory, $x \mapsto \frac{d\tau}{d\nu}(\varphi(x))$ is called the “likelihood function”.

³⁰ From expression (44) it follows that if x_1, x_2, \dots is a collection of (independent) random sample elements of the prior probability measure π_{prior} , and if, for every element, a random decision is taken to conserve or discard it with the probability of being conserved equal to $k \frac{d\tau}{d\nu}(\varphi(x_i))$ (where the positive constant k is arbitrary, excepted that it must ensure that the maximum attained value is ≤ 1), then, the collection x'_1, x'_2, \dots , of conserved elements is a sample of π_{post} (Mosegaard and Tarantola [1995]). And (as it follows from the definition of image of a measure) the collection $\varphi(x'_1), \varphi(x'_2), \dots$ is a sample of τ_{post} .

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