Probabilistic Mappings of Probability Measures

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Inverse Days, Tahkovuori, Finland, December 2008
The Bayes-Popper approach.

Using observations to infer the values of some parameters corresponds to solving an inverse problem. Practitioners sometimes seek the best solution implied by the data, but observations should only be used to falsify possible solutions, not to deduce any particular solution.
$x \rightarrow y = \varphi(x)$
\[
x \rightarrow y = \varphi(x)
\]

\[
x \rightarrow \theta_x(y)
\]
**Intersection of measures:** Given a measure space \((\Omega, \mathcal{F}, \mu)\), let \(\mu_1\) and \(\mu_2\) be two measures such that, for every \(F \in \mathcal{F}\), the following expressions make sense:

\[
\mu[\mu_1, \mu_2][F] = \int_{\omega \in F} \frac{d\mu_1}{d\mu}(\omega) \frac{d\mu_2}{d\mu}(\omega) \, d\mu(\omega)
\]

\[
(\mu_1 \cap \mu_2)[F] = \frac{\mu[\mu_1, \mu_2][F]}{\mu[\mu_1, \mu_2][\Omega]}
\]

The probability measure \((\mu_1 \cap \mu_2)\) is the *intersection* of the two measures \(\mu_1\) and \(\mu_2\). The quintuplet \(\{\Omega, \mathcal{F}, \mu, \mu_1, \mu_2\}\) is a finite *Radon-Nikodym space*. 

Family of probability measures: Let \((X, \mathcal{F}_X)\) and \((Y, \mathcal{F}_Y)\) be two measurable spaces, and assume that to every \(x \in X\) it is associated a probability measure \(\theta_x\) on \((Y, \mathcal{F}_Y)\). We use the notation \(\Theta\) for the set \(\{\theta_x \mid x \in X\}\), and we say that \(\Theta\) is a family of probability measures from \(X\) on \((Y, \mathcal{F}_Y)\).
Image: Let \((X, \mathcal{F}_X)\) and \((Y, \mathcal{F}_Y)\) be two measurable spaces, \(\Theta = \{\theta_x \mid x \in X\}\) a family of probability measures from \(X\) on \((Y, \mathcal{F}_Y)\), and \(\pi_X\) a measure on \((X, \mathcal{F}_X)\). If for any \(F_Y \in \mathcal{F}_Y\), the function \(\theta_x[F_Y]\) is \(\pi_X\)-measurable, the image of \(\pi_X\) (by the family \(\Theta\)) is the measure on \((Y, \mathcal{F}_Y)\), denoted \(\Theta[\pi_X]\), defined by the condition

\[
(\Theta[\pi_X])[F_Y] = \int_X \theta_x[F_Y] \, d\pi_X(x) \quad \text{for every } F_Y \in \mathcal{F}_Y.
\]

Later on the following notation

\[
d(\Theta[\pi])(y) = \int_X d\theta_x(y) \, d\pi(x)
\]

shall be used for this definition.
$x \rightarrow \theta_x(y)$

**image of a probability distribution**
**Reciprocal image:** Let \((X, \mathcal{F}_X, \mu_X)\) and \((Y, \mathcal{F}_Y, \mu_Y)\) be two measure spaces, \(\pi_Y\) a measure on \((Y, \mathcal{F}_Y)\), and \(\Theta = \{\theta_x \mid x \in X\}\) a family of probability measures from \(X\) on \((Y, \mathcal{F}_Y)\) such that for every \(x \in X\), \((Y, \mathcal{F}_Y, \mu_Y, \theta_x, \pi_Y)\) is a Radon-Nikodym space. The *reciprocal image* of \(\pi_Y\) (by the family \(\Theta\)), denoted \(\Theta^{-1}[\pi_Y]\), is the measure on \((X, \mathcal{F}_X)\), absolutely continuous w.r.t. \(\mu_X\), defined, via its \(\mu_X\)-density, as

\[
\frac{d(\Theta^{-1}[\pi_Y])}{d\mu_X}(x) = \int_Y (\theta_x \cap \pi_Y)(y) \, d\mu_Y(y) \quad .
\]

\[
\frac{d(\Theta^{-1}[\pi_Y])}{d\mu_X}(x) = \mu_Y[\theta_x, \pi_Y][Y] \quad .
\]
reciprocal image of a probability distribution

\[ x \rightarrow \theta_x(y) \]
**Product of measures:** Given two measurable spaces \((X, \mathcal{F}_X)\) and \((Y, \mathcal{F}_Y)\), to every pair of measures \(\tau_X\) and \(\tau_Y\), respectively on \((X, \mathcal{F}_X)\) and on \((Y, \mathcal{F}_Y)\), is associated a measure \(\tau_X \times \tau_Y\) on \((X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)\), called the *product measure*, defined for every \(F_X \in \mathcal{F}_X\) and every \(F_Y \in \mathcal{F}_Y\), by

\[
(\tau_X \times \tau_Y)[F_X \times F_Y] = \int_{F_X} d\tau_X(x) \int_{F_Y} d\tau_Y(y) .
\]

We shall use the notation

\[
d(\tau_X \times \tau_Y)(x, y) = d\tau_X(x) \times d\tau_Y(y) .
\]
Marginal measures: Given two measurable spaces \((X, \mathcal{F}_X)\) and \((Y, \mathcal{F}_Y)\), to every measure \(\pi\) on \((X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)\) are associated the two marginal measures (respectively on \((X, \mathcal{F}_X)\) and on \((Y, \mathcal{F}_Y)\)) defined (respectively for every \(F_X \in \mathcal{F}_X\) and every \(F_Y \in \mathcal{F}_Y\)) by

\[
\pi_X[F_X] = \pi[F_X \times Y] \quad ; \quad \pi_Y[F_Y] = \pi[X \times F_Y].
\]
**Inference space:** Given two measure spaces \((X, \mathcal{F}_X, \mu_X)\) and \((Y, \mathcal{F}_Y, \mu_Y)\), given a family \(\Theta = \{\theta_x \mid x \in X\}\) of probability measures on \((Y, \mathcal{F}_Y)\), and given a probability measure \(\pi\) on \((X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)\), an *inference space* is the quintuplet \(\mathcal{I} = \{X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \pi, \Theta\}\).
Premise inference space: If an inference space \( I = \{ X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \pi, \Theta \} \) is such that

\[
\pi = \pi_X \times \pi_Y ,
\]

where \( \pi_X \) and \( \pi_Y \) are the two marginal probability measures of the probability measure \( \pi \), if the intersections \( \theta_x \cap \pi_Y \) are nonempty (for all \( x \in X \)), we say that \( I \) is a premise inference space.
**Conclusion inference space:** If an inference space $\mathcal{I} = \{ X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \pi, \Theta \}$ is such that

$$\Theta[\pi_X] = \pi_Y,$$

where $\pi_X$ and $\pi_Y$ are the two marginal probability measures of the probability measure $\pi$, we say that $\mathcal{I}$ is a *conclusion inference space*. 
Theorem: If $\mathcal{I} = \{ X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \pi, \Theta \}$ is a premise inference space, then the quintuplet $\mathcal{I} = \{ X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \tilde{\pi}, \tilde{\Theta} \}$ where the probability measures of the family $\tilde{\Theta} = \{ \tilde{\theta}_x \mid x \in X \}$ are defined as

$$\tilde{\theta}_x = \theta_x \cap \pi_Y$$

and where the probability measure $\tilde{\pi}$ is defined by

$$\tilde{\pi}[F_X \times F_Y] = \int_{F_X} \int_{F_Y} \tilde{\theta}_x(y) \ d\mu_Y(y) \ d\pi_X(x),$$

is a conclusion inference space, i.e. the two marginal probability measures of $\tilde{\pi}$, say $\tilde{\pi}_X$ and $\tilde{\pi}_Y$, are related as

$$\tilde{\Theta}[\tilde{\pi}_X] = \tilde{\pi}_Y.$$
Furthermore, one has

\[ \tilde{\pi}_X = \pi_X \cap \Theta^{-1}[\pi_Y] \quad \text{and} \quad \tilde{\pi}_Y = \pi_Y \cap \Theta[\pi_X] \]
Theorem: If $\mathcal{I} = \{X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \pi, \Theta\}$ is a premise inference space, then the quintuplet $\mathcal{I} = \{X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \tilde{\pi}, \tilde{\Theta}\}$ where the probability measures of the family $\tilde{\Theta} = \{\tilde{\theta}_x \mid x \in X\}$ are defined as

\[\tilde{\theta}_x = \theta_x \cap \pi_Y\]

and where the probability measure $\tilde{\pi}$ is defined by

\[\tilde{\pi}[F_X \times F_Y] = \int_{F_X} \int_{F_Y} \tilde{\theta}_x(y) d\mu_Y(y) d\pi_X(x),\]

is a conclusion inference space, i.e. the two marginal probability measures of $\tilde{\pi}$, say $\tilde{\pi}_X$ and $\tilde{\pi}_Y$, are related as

\[\tilde{\Theta}[\tilde{\pi}_X] = \tilde{\pi}_Y.\]
$x \rightarrow \theta_x(y)$
\[ x \rightarrow \theta_x(y) \]
$x \mapsto \theta_x(y)$

$\varphi_1 \quad r_1$

$\varphi_2 \quad r_2$

$\varphi_3 \quad r_3$

$\varphi_1 \quad \tilde{r}_1$

$\varphi_2 \quad \tilde{r}_2$

$\varphi_3 \quad \tilde{r}_3$
The End.