Emission Coordinates as Positioning System (I)
• 2D-Positioning: a qualitative description.
• Inertial null coordinates.
• The metric in null (emission) coordinates.
• Kinematics in null coordinates.
In a four dimensional space-time our positioning system is determined by four clocks broadcasting their proper time (principal emitters). In 2D we have two emitters $\gamma_1(\tau^1)$ and $\gamma_2(\tau^2)$ (blue lines in fig. 1.1). Every emitter $\gamma_i$ broadcasts his proper time $\tau^i$. The future light cones (dark and light red lines, respectively, in fig. 1.1) cut in the region between both emitters and they coincide outside.
The internal region and the emitters world lines define a coordinate domain (pink area in fig. 1.2). Every event on this domain can be distinguished by the times \((\tau^1, \tau^2)\) received from the emitter clocks. In other words, the past light cone of every event on this domain cut the emitter world lines at \(\gamma_1(\tau^1)\) and \(\gamma_2(\tau^2)\) respectively. Then \((\tau^1, \tau^2)\) are the emission coordinates of this event. On the contrary, an observer outside this area receives the same times \((\tau^1, \tau^2)\) that the next principal observer at the emission event.

Let $\gamma$ be an observer equipped with a receiver which allows the reception of the proper times $(\tau^1, \tau^2)$ at each point of his trajectory. Then, this observer knows his trajectory in these emission coordinates. It is then a user of this positioning system. A user could also carry a clock measuring his proper time $\tau$ (fig. 1.3). Finally, an emission positioning system could be performed with another quality (autonomous): the principal emitters are transmitters, so that, users which receive the signals and broadcast them together with their proper time (fig. 1.4).
Inertial null coordinates

In a flat 2-dimensional space-time we can associate the inertial null coordinates \( \{u, v\} \) with every inertial coordinate system \( \{t, x\} \) (fig. 2.1):

\[
\begin{align*}
  u &= t + x \\
  v &= t - x \\
  t &= \frac{1}{2}(u + v) \\
  x &= \frac{1}{2}(u - v)
\end{align*}
\]

In inertial null coordinates, the metric tensor takes the form:

\[
ds^2 = dt^2 - dx^2 = du \, dv, \quad g = \frac{1}{2}du \otimes dv
\]

So, its covariant and contravariant components are, respectively:

\[
(g_{ij}) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}
\]

It is well known that a boost between two inertial systems \( \{t, x\}, \{\bar{t}, \bar{x}\} \) with a relative velocity \( \beta = \tanh \psi \) is given by

\[
\begin{align*}
  \bar{t} &= \cosh \psi \, t + \sinh \psi \, x \\
  \bar{x} &= \sinh \psi \, t + \cosh \psi \, x
\end{align*}
\]
Inertial null coordinates

But it is not so known that, in inertial null coordinates a boost takes the simple expression (fig. 2.2):

\[ \bar{u} = e^{\psi} u \]
\[ \bar{v} = e^{-\psi} v \]

Let us note that the factor

\[ s = e^{\psi} = \sqrt{\frac{1 + \beta}{1 - \beta}} = 1 + z \]

is the shift parameter between both inertial systems.

The boost (1) gives the internal transformation between inertial null coordinates. In inertial null coordinates the invariance of the Minkowskian interval states

\[ \bar{u}\bar{v} = uv \]
The metric in null (emission) coordinates

For any bidimensional space-time, there exist some coordinates in which the metric has the form

\[ ds^2 = m(u, v) \, du \, dv \]

\[ (g_{ij}) = \frac{m}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

The contravariant metric has then the components

\[ (g^{ij}) = \frac{2}{m} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

For a flat bidimensional space-time, the component is factorizable:

\[ m(u, v) = U(u)V(v) \]

so that there always exists a change of coordinates

\[ \bar{u} = U(u), \quad \bar{v} = V(v) \quad \text{for which} \quad ds^2 = U(u)V(v) \, du \, dv = d\bar{u} \, d\bar{v} \]
The metric in null (emission) coordinates

...
Kinematics in null coordinates

Now we will consider some basic and simple kinematic results and write them in a given null system \{u, v\}. Later we will apply all these results to the emitter null coordinates.

In terms of his proper time $\tau$, the trajectory of an observer $\gamma$ is (fig. 4.1):

$$
u = u(\tau)$$

$$
v = v(\tau)$$

and its tangent vector:

$$T(\tau) = (T^u, T^v) = (\dot{u}(\tau), \dot{v}(\tau))$$

where, in what follows, a dot means derivative with respect proper time. The unit condition for $T$ connects the metric component with the observer trajectory:

$$m(u(\tau), v(\tau)) = \frac{1}{\dot{u}(\tau)\dot{v}(\tau)}$$

This simple relation plays an important role in 2D-positioning as we will see later. It can be read as follows: if one knows the unit tangent vector of an observer in terms of his proper time, then one knows the metric on the trajectory of this observer.
Kinematics in null coordinates

From one of the two functions (3) we can obtain the proper time of the observer $\gamma$. For example, we have:

$$
\tau = \tau(u) = u^{-1}(u)
$$

Moreover the trajectory is given by a function $v = F(u)$ that can be obtained as (fig. 4.2):

$$
v = F(u) = v(\tau(u))
$$

Then the unit condition can be written:

$$
[\tau'(u)]^2 = m(u, F(u))F'(u)
$$
Kinematics in null coordinates

From here two statements follow:

- If the trajectory \( v = F(u) \) of an observer \( \gamma \) is known, then one knows the metric on the trajectory iff one knows the \( \gamma \)-proper time \( \tau = \tau(u) \) (fig. 4.2).

- If the metric component \( m(u, v) \) is known, then the trajectory of the observers \( \gamma \) with a fixed proper time \( \tau = \tau(u) \) is given by the solutions \( v = F'(u) \) of the first order differential equation (4).

Evidently, a trajectory with a fixed proper time \( \tau = \tau(u) \) passes for any fixed event in the coordinate domain (fig. 4.3, 4.4).
Emission Coordinates as Positioning System (II)

The tridimensional case
CONTENTS

• Comparison of the tridimensional case with the bidimensional case.
  – Elements dependents on the spacetime and elements dependents on the satellites.
  – Coordinate lines and hipersurfaces: causal character of coordinate vectors and forms.
  – Unicity of the event determined by the grid parameters: “interior” and “exterior” zones.

• Grid parameters and $\tau$ coordinates.
  – Differentiability and injectivity.
  – The unicity frontier and the four domain of coordinates.
• The image of the space-time in the grid of parameters.
  – The position of the emitters in the grid.
  – The shadows in the grid: Limits from causality.
  – Examples for static emitters in Minkowski.
    * Light-like future infinity.
    * The inertial space.

• The contravariant metric. The element of volume and degeneracy of the metric.
  – The Jacobian of the transformation for static emitters in Minkowski.

• The covariant metric. Causal character of the coordinate vectors.
Comparison of 3D with 2D

The spacetime determines for each event:

2D
2 unique null directions, independently of the trajectory of the satellites.

3D
a past null cone, but the 3 directions of the signals depend on the trajectory of the satellites.
Comparison of 3D with 2D

The grid is determined by:

2D
The spacetime determines the coordinate hypersurfaces and the satellites determines their numbering and their domain.

3D
It is determined inseparably by both, the space-time and the trajectories of the satellites.
Comparison of 3D with 2D

The event is uniquely determined:

**2D**

in the interior zone between the two satellites.

**3D**

in a kind of unbounded ‘interior’ zone.

For Minkowski, in the ‘exterior zone’ each set of parameters correspond to two events.
Exemple of two intersections in the ‘exterior’ zone
Exemple of two intersections in the ‘exterior’ zone
Comparison of 3D with 2D

Causal character of coordinate vectors and forms

Coordinate hypersurfaces: \( \tau^A = \text{ctant} \), represent the coordinate 1-forms \( d\tau^A \). They are null.

Coordinate lines: \( \tau^B = \text{ctant} \ \ \forall B \neq A \), are the integral curves of the coordinate vectors \( \partial_A \). They are not time-like:
The emission parameters of an space-time point will be simply the 3 values $\tau^1, \tau^2, \tau^3$ that are received (or could be received) at this point from the 3 emitters $S_1, S_2, S_3$ broadcasting their proper time.

To label the points in a tridimensional space-time, we need three co-ordinates.

The 3 scalar fields provided by the emission parameters $\tau^A$ are good candidates for being coordinates, but we should check some required properties.

- Continuity and Differentiability
- Non-degeneration
Continuity and Differentiability

Given a space-time with a metric two times differentiable, if the word-line of the emitter $S_A$ is differentiable then the scalar field $\tau^A$ is continuous and differentiable, except at the word-line of the emitter itself, where it is not differentiable.

This is true if we consider only regions of the space-time sufficiently small or with sufficiently weak gravitational field, for conserving the topology of the null cone of Minkowski:

We will not consider at all the existence of conjugated points, where the same signal reach the same event through two different paths.

This should not suppose any restriction in any practical application around the Earth or in the Solar System.
Emission parameters and emission coordinates

Non-degeneration

The system of parameters \( \{ \tau^A \} \) will be degenerate at the points where the three covectors \( d\tau^A \) are linearly dependent:

\[
d\tau^1 \wedge d\tau^2 \wedge d\tau^3 = 0.
\]

When can this condition be satisfied?

The space-time has Lorentzian signature. Thus, at any point, any bidimensional plane will contain none, one or two null directions. The covectors \( d\tau^A \) are light-like. Hence, they are contained in a 2-plane only when two of them have the same direction.

This happens only when two of the null cones, \( \tau^A = \text{const.} \), are tangent: where two of the light-like geodesics followed by the signals coincide.

The emission parameters are degenerate at any of the shadows that an emitter \( S_A \) makes to another \( S_B \): \( S_{AB} \).
Emission parameters and emission coordinates

Domains of coordinates

For 3 emitters there are 6 shadows: 2 for each emitter. The two shadows of each emitter start from the emitter itself and extend to the infinite in different directions.

Thus, they separate the region between the two shadows from the region outside the two shadows.

Considering this separation for each of the 3 emitters, this gives at least 4 disconnected regions delimited by the shadows.

Inside each of these regions, the emission parameters are non-degenerate. Therefore, they constitute actually emission coordinates in each of these domains.

We have then 4 different domains of emission coordinates. The union of the 4 domains occupies the whole space-time except the shadows and the world-lines of the emitters.
Inside each domain the map from the space-time to the grid of emission parameters is evidently injective.

At the shadows the parameters are degenerate but they are still continuous. Thus, each pair of domains of emission coordinates matches continuously in the shadow which separates them.

Can the emission parameters coordinate the shadow?

The answer is affirmative, provided that the shadow does not cross with another shadow.
Emission parameters and emission coordinates

The continuity through the shadows. Injectivity

The word-line of the emitter $S_1$ is a time-like curve that can be parametrized by its proper time $\tau^1$:

$$\{\tau^1, \tau^2 = f^2_1(\tau^1), \tau^3 = f^3_1(\tau^1)\}$$

The shadow $S_{12}$ of $S_1$ to $S_2$ is characterized by having the same emission parameters $\tau^1$ and $\tau^2 = f^2_1(\tau^1)$ as the word-line of $S_1$. The third parameter is free except for the inequality $\tau^3 > f^3_1(\tau^1)$, imposed by the condition of being propagated to the future and not to the past.

$$\{\tau^1, \tau^2 = f^2_1(\tau^1), \tau^3 > f^3_1(\tau^1)\}$$

Thus, we can essay to coordinate the shadow by the two parameters $\{\tau^1, \tau^3\}$, where $\tau^1$ determine the null geodesic and $\tau^3$ determine a point in each null geodesic in the shadow.
Emission parameters and emission coordinates

The continuity through the shadows. Injectivity

They will be good coordinates if they are not degenerate, that is if the two covectors $d\tau^1|_{S_{12}}$ and $d\tau^3|_{S_{12}}$ projected in the surface $S_{12}$ are not linearly dependent.

In the surface, the direction of $d\tau^1|_{S_{12}}$ is given by the null geodesic $\tau^1 = \text{const}$. If the direction of $d\tau^3|_{S_{12}}$ coincides with $d\tau^1|_{S_{12}}$ at some point, then $d\tau^1$ and $d\tau^3$ in the space-time will contain the same null direction.

$d\tau^1$ and $d\tau^3$ are given by surfaces of null cones, Hence, they can coincide in a null direction at some point only if they are tangent, that is, if they share the corresponding null geodesic.

Therefore, $\{\tau^1, \tau^3\}$ are good coordinates for the shadow of $S_1$ to $S_2$ except at the geodesics which also contain the third emitter $S_3$, that is, where the shadow cross with another shadow.
Emission parameters and emission coordinates

This result has two interesting consequences:

- Although no domain of emission coordinates contains the shadows, the emission parameters determine uniquely our position in the shadows.

- The shadows are surfaces in both, the space-time and the grid of emission parameters. In the grid, each of the 4 domains of coordinates are bounded by the surface image of the shadows.

The image of the space time in the grid of emission parameters is continuous. Hence, for each pair of domains of coordinates separated by a shadow, there appear two possibilities in grid:

- Each one is at a different side of the image of the shadow,
- The two images are superposed both at the same side of the image of the shadow.
The image of the space-time in the grid

The emission parameters \( \{ \tau^A \} \) defines a map from the space-time to \( \mathbb{R}^3 \). This \( \mathbb{R}^3 \), image space of the emission parameters, will be called the grid of parameters.

Which is the image of the space-time in the grid?

To represent the grid we will utilize the usual perspective and projections proper to the Euclidean \( \mathbb{R}^3 \). We must keep in mind that this is only a convenient representation, which has no relation with the metric properties of the space-time.
The image of the space-time in the grid

World-lines: future-oriented time-like curves

The word-line of any punctual object or observer is a time-like curve oriented to the future, that is, a curve $c(\lambda)$, whose tangent vector $u = \frac{d}{d\lambda} c(\lambda)$ is time-like and whose parameter $\lambda$ is increasing towards the future.

The word-line of each of the emitters $S_A$, is future-oriented. Hence, the null cones given by $\tau^A = \text{const.}$ are ordered so that, given any two of them, the one with greater $\tau^A$ is in the causal future of the other. Hence, every word-line $c(\lambda)$ will cross the surfaces $\tau^A = \text{const.}$ in increasing order. The image of any word-line in the grid will have the form

$$c(\lambda) \mapsto (c^1(\lambda), c^2(\lambda), c^3(\lambda))$$

where the three functions $c^A(\lambda)$ are all increasing monotonic:

$$\frac{d}{d\lambda} c^A(\lambda) > 0 .$$
The image of the space-time in the grid

The trajectory of the emitters

The word-line of each of the emitters $S_A$ can be parametrized with its own proper time $\tau^A$. Thus, the image of them in the grid of emission parameters will be of the form

$$
S_1 \mapsto (\tau^1, f_1^2(\tau^1), f_1^3(\tau^1))
$$

$$
S_2 \mapsto (f_2^1(\tau^2), \tau^2, f_2^3(\tau^2))
$$

$$
S_3 \mapsto (f_3^1(\tau^3), f_3^2(\tau^3), \tau^3)
$$

where all the functions $f^B_A(\tau^A)$ are increasing monotonic:

$$
\frac{d}{d\tau^A} f^B_A(\tau^A) > 0 \quad \forall A, B.
$$
The image of the space-time in the grid

The shadows and the bounds from causality

The surfaces image of the shadows in the grid limits the region of the grid where any point of the space-time can be mapped. This conclusion follows by considering causality conditions.

Take the emitter $S_1$ when its clock reads $\tau^1_0$. Its position in the grid will be $(\tau^1_0, f^2_1(\tau^1_0), f^3_1(\tau^1_0))$. The emitter is at that moment in the null cone $\tau^2 = f^2_1(\tau^1_0)$, emitted by $S_2$ when its clock reads $\tau^2_0 = f^2_1(\tau^1_0)$.

Hence, $S_1(\tau^1_0)$ is in the causal future of $S_2(\tau^2_0)$:

$$S_1(\tau^1_0) \in \text{Future} \left[ S_2(\tau^2_0) \right]$$

The condition of causality is defined as the causal partial order given by the transitivity property

$$A \in \text{Future} \left[ B \right] \text{ and } B \in \text{Future} \left[ C \right] \Rightarrow A \in \text{Future} \left[ C \right].$$
The image of the space-time in the grid

The shadows and the bounds from causality

Any point of the grid with $\tau^1 > \tau_0^1$ will correspond to an space-time point in the future of $S_1(\tau_0^1)$, independently of the other grid coordinates.

$$\tau^1(p) > \tau_0^1 \iff p \in \text{Future}[S_1(\tau_0^1)]$$

Any point of the grid with $\tau^2 < \tau_0^2$ will correspond to an space-time point not in the future of $S_2(\tau_2^2)$.

$$\tau^2(p) < \tau_0^2 \iff p \notin \text{Future}[S_2(\tau_0^2)]$$

Applying the causality condition we obtain

$$p \in \text{Future}[S_1(\tau_0^1)] \Rightarrow p \in \text{Future}[S_2(\tau_0^2)] .$$
The image of the space-time in the grid

The shadows and the bounds from causality

Hence, any point \((\tau^1, \tau^2, \tau^3)\) of the grid satisfying \(\tau^2 > f_1^2(\tau^1)\) does not correspond to any point of the space-time. The general result for any two emitters is the following:

\[
\forall p \in \text{Space-time} \quad \text{so that} \quad \tau^B(p) > f_A^B(\tau^A(p)),
\]

This result answers the question made before. Since the image of the shadows delimit the permitted region in the grid, thus the image of two domains of emission coordinates separated by a shadow must be at the same side of the image of this shadow. Hence, they will be superposed.
Static emitters in Minkowski

The future light-like infinite in the grid

A null geodesic in Minkowski is a curve of the form

\[ p(\lambda) = p_0 + \lambda \ell = (t_0 + \lambda)e_0 + \vec{p}_0 + \hat{n}, \]

where \( p_0 = \vec{p}_0 + t_0e_0 \) is any event, \( \lambda \in \mathbb{R} \) is the parameter of the curve, and \( \ell = \hat{n} + e_0 \) is the null direction of the geodesic.

Their emission coordinates are

\[ \tau^A = \lambda + t_0 - |\lambda\hat{n} + \vec{p}_0 - \vec{s}_A| \]

The future infinite limit correspond to make \( \lambda \to \infty \). This limit gives

\[ \lim_{\lambda \to \infty} \tau^A = \lim_{\lambda \to \infty} \lambda + t_0 - \lambda \left| \hat{n} + \frac{1}{\lambda}(\vec{p}_0 - \vec{s}_A) \right| = t_0 + \vec{s}_A \cdot \hat{n}. \]

Observe that the first two terms are the same for the three coordinates.
We have just seen that the future null infinity is an elliptic cylinder in the grid.

This implies that the image in the grid of the space \((t = \text{const.})\) cannot reach this cylinder, since, the space-like infinity is in the past of the future null infinity.

Hence, the coordinate \(\mu^3\) of the space-like infinity must diverge towards \(-\infty\).
The metric in emission coordinates

The contravariant metric

Once we have seen some general properties about the global behaviour of emission coordinates, we can study more specifically the local metrical properties.

The covectors $d\tau^A$ are light-like:

$$d\tau^A \cdot d\tau^A \equiv g^*(d\tau^A, d\tau^A) = 0.$$ 

That is, the diagonal components of the contravariant metric in emission coordinates are zeros:

$$(g^{AB}) = \begin{pmatrix} 0 & g^{12} & g^{13} \\ g^{12} & 0 & g^{23} \\ g^{13} & g^{23} & 0 \end{pmatrix}.$$ 

For signature $(+, -, -)$, $d\tau^A \cdot d\tau^B = g^{AB} > 0 \quad \forall A \neq B$. 

The metric in emission coordinates

The element of volume and the degeneracy of the metric

We have just found that the contravariant metric in emission coordinates is of the form

\[
(g^{AB}) = \begin{pmatrix}
0 & g^{12} & g^{13} \\
g^{12} & 0 & g^{23} \\
g^{13} & g^{23} & 0 \\
\end{pmatrix}, \quad \text{with} \quad g^{AB} > 0 \quad \forall \, A \neq B.
\]

In order to have a good coordinate system, the metric must be non-degenerate. That is, its determinant must not vanish:

\[
\det(g^{AB}) = 2g^{12}g^{13}g^{23} \neq 0.
\]

Hence, as we expected, the metric will be degenerate only when two of the covectors, for instance \(d\tau^1\) and \(d\tau^2\), are parallel, giving \(g^{12} = 0\).

Observe that the determinant is positive, \(\det(g^{AB}) > 0\), what is in agreement with the signature chosen.
The element of volume and the degeneracy of the metric

The element of volume in emission coordinates is

\[ dV = \frac{\pm 1}{\sqrt{\det(g^{AB})}} \, d\tau^1 \wedge d\tau^2 \wedge d\tau^3 = \frac{\pm 1}{\sqrt{2g^{12}g^{13}g^{23}}} \, d\tau^1 \wedge d\tau^2 \wedge d\tau^3. \]

The sign depends on the orientation chosen for the space-time. But, in fact, it change when we pass from one domain of coordinates to a contiguous one. If, for instance, we choose the positive sign in the interior domain, then the sign will be negative in all the 3 exterior domains delimited by the 2 shadows of each of the 3 emitters.

This fact is a consequence of the result found before, since the ‘reflection’ of the emission parameters in the shadow implies that the orientation (the Jacobian) of the map from the space-time to the grid is inverted in passing from one domain to another.
The Jacobian of the change of coordinates Cartesian $\rightarrow$ Emission

In cartesian coordinates, the volume elements has the simple form $dV = dx^0 \wedge dx^1 \wedge dx^2$.

The relation is given by the Jacobian of the transformation from Cartesian coordinates to emission coordinates:

$$\left| \frac{\partial \tau^A}{\partial x^\mu} \right| = \frac{d\tau^1 \wedge d\tau^2 \wedge d\tau^3}{dx^0 \wedge dx^1 \wedge dx^2} = 4 \sin \frac{\theta^{12}}{2} \sin \frac{\theta^{23}}{2} \sin \frac{\theta^{31}}{2}$$

The Jacobian is greater at the interior of the triangle formed by the emitters than at the exterior. Indeed it changes abruptly when we go out of the triangle.

The value at the center, for the case of the emitters forming an equilateral triangle, is the maximum value for the Jacobian:

$$J = 4 \left( \sin \frac{\pi}{3} \right)^3 = \frac{3}{2} \sqrt{3}$$
The metric in emission coordinates

The covariant metric

Given the contravariant metric $g^{AB}$, the covariant metric is obtaining simply by computing its inverse:

$$
(g_{AB}) = \frac{1}{2g^{12}g^{13}g^{23}} \begin{pmatrix}
-(g^{23})^2 & g^{13}g^{23} & g^{12}g^{23} \\
g^{13}g^{23} & -(g^{13})^2 & g^{12}g^{13} \\
g^{12}g^{23} & g^{12}g^{13} & -(g^{12})^2
\end{pmatrix}
$$

$$
= \frac{1}{2} \begin{pmatrix}
-g^{23} & 1 & 1 \\
g^{12}g^{13} & -g^{13} & 1 \\
g^{12} & g^{13} & -g^{12}g^{23}
\end{pmatrix}
$$

Note the simplicity of the components off the diagonal,

$$
g_{AB} = \frac{1}{2g^{AB}} \quad \forall \ A \neq B .
$$
The metric in emission coordinates

The covariant metric

The diagonal components are negative. This means that the coordinate vectors are space-like,

$$\partial_A \cdot \partial_A = g_{AA} < 0 ,$$

confirming what we have discussed about the causal character of the coordinate vectors by considering the intersections of the null cones.