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ANALYTICAL SOLUTIONS OF LOVE NUMBERS FOR A HYDROSTATIC ELLIPSOIDAL INCOMPRESSIBLE HOMOGENEOUS EARTH

MARIANNE GREFF-LEFFTZ[1], LAURENT MÉTIVIER[1] and HILAIRE LEGROS[2]
[1]*Institut de Physique du Globe de Paris, 4 place Jussieu, 75252 Paris 05, France, e-mail: greff@ipgp.jussieu.fr*
[2]*E.O.S.T., 5 rue René Descartes, 67084 Strasbourg, France*

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Abstract. Tidal forces acting on the Earth cause deformations and mass redistribution inside the planet involving surface motions and variation in the gravity field, which may be observed in geodetic experiments. Because for space geodesy it is now necessary to achieve the mm level in tidal displacements, we take into account the hydrostatic flattening of the Earth in the computation of the elasto-gravitational deformations. Analytical solutions are derived for the semi-diurnal tides on a slightly elliptical homogeneous incompressible elastic model. That simple analytical Earth's model is not a realistic representation of any real planet, but it is useful to understand the physics of the problem and also to check numerical procedures. We rediscover and discuss the Love's solutions and obtain new analytical solutions for the tangential displacement. We extend these analytical results to some geodetic responses of the Earth to tidal forces such as the perturbation of the surface gravity field, the tilt and the deviation of the vertical with reference to the Earth's axis.

Key words: body tides, elasto-gravitational deformations, Love numbers

1. Introduction

Tidal forces acting on the Earth cause deformations and mass redistribution inside the planet involving surface motions and variation in the gravity field, which may be observed in geodetic experiments.

The tide is one of the most important external source and consequently one of the most studied. In 1862, Lord Kelvin made the first calculus of the elastic deformation of an homogeneous incompressible Earth under the action of the tidal gravitational potential. Some years latter, Love (1911) studied a compressible homogeneous Earth's model and showed that the tidal effects could be represented by a set of dimensionless numbers, the so-called Love numbers. Today, body tides effects at and outside the Earth's surface are always modeled in terms of the tidal Love numbers noted k, h, and l for, respectively, the gravitational field, the radial and tangential displacement. Takeuchi (1950) obtained a first estimation of these Love numbers by a numerical integration of the equations using a reference Earth's model given by seismologists. These results have then been extended by Wahr and Bergen (1986) and Dehant (1987) to an anelastic mantle.

The effect of the spheroidal shape of the Earth on the body tides has been already estimated by several authors (Wahr, 1981; Dehant, 1987, 1991, 1995; Buffett et al., 1993; Wang, 1994; Mathews et al., 1995).

An analytical solution for this effect is only possible for a simple model. Love (1911) derived analytical solution, using an Eulerian formulation, for the semi-diurnal tides on a slightly elliptical homogeneous incompressible elastic model. In this paper, we would like to rediscover the Love's solutions (in term of surface mass redistribution potential and radial displacement) using a Lagrangian formulation and obtain new analytical solutions for the tangential displacement. We intend to extend these results to some geodetic responses of the Earth to tidal forces such as the perturbation of the surface gravity field (with the associated gravimetric factor), the tilt and the deviation of the vertical with reference to the Earth's axis.

That simple Earth's model is not a realistic representation of any real planet, but because it allows analytical solutions, it is useful to understand the physics of the problem, and especially the influence of each geometric and physical parameter, and also to check the numerical procedure of a new method developed in a next paper (Métivier et al., 2005), This paper is organized as follows. In the first part (Section 2), we recall the classical elasto-gravitational theory for a spheroidal hydrostatic pre-stress planet using the formalism proposed by Dahlen (1968) (see also Smith, 1974; Dahlen, 1976). In Section 3, we present the analytical solutions for an incompressible homogeneous elastic planet submitted to the semi-diurnal tidal potential and discuss our solutions with respect to the Love' ones. We extend these results (in Section 4) to the analytical study of the geodetic and gravimetric responses of the Earth to the tides on the spheroidal deformed surface.

2. Elasto-gravitational Theory for a Spheroidal Hydrostatic Pre-stress Planet

To describe the motion of slightly elliptical elastic Earth, we use the formalism proposed by Dahlen and Tromp (1998). The rotating planet is submitted to luni-solar gravitational forces $\rho \vec{f}$ with $\vec{f} = \vec{\nabla} V$, where V is the luni-solar tidal potential. Under the effects of this cause, the Earth is deformed. The state of an internal particle is described by its density ρ, its displacement $\vec{u}$, its strain tensor $\left[\epsilon_{ij} = \frac{1}{2}([\partial_i u_j + \partial_j u_i])\right]$, its stress tensor $\sum_{ij}$, its potential carried on by the rest of the body Φ and the tidal potential V. We note Ψ the centrifugal potential:

$$\Psi = \Psi_o \left[1 - \frac{3\cos^2\theta - 1}{2}\right] \quad \text{with} \quad \Psi_o = \frac{\Omega^2 r^2}{3}, \tag{1}$$

where Ω is the uniform angular velocity, r the radius, and θ the colatitude. The mass and impulsion conservation, and the Poisson equation are written:

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{v} = 0, \\ &\rho\frac{\mathrm{d}^2\vec{u}}{\mathrm{dt}^2} = \vec{\nabla}\cdot\overline{\overline{\Sigma}} + \rho\vec{\nabla}(\Phi + \Psi) + \rho\vec{\nabla}V, \\ &\Delta\Phi = -4\pi G_\rho, \end{aligned} \tag{2}$$

where $\vec{v}$ is the velocity. In our hypotheses, these are the equations of the problem. We have to add a rheological law to link the stress tensor to the strain tensor.

The fundamental assumption in the elasto-gravitational theory is that the deformations are small in comparison with the reference configuration in hydrostatic equilibrium. We can thus use a perturbations theory. We introduce, in a point of the volume, the Eulerian perturbations. We put:

$$\begin{aligned} &\rho = \rho_{\text{ref}} + \rho_1^e, \\ &\overline{\overline{\Sigma}} = \overline{\overline{\Sigma_{\text{ref}}}} + \overline{\overline{\Sigma_1^e}}, \\ &\Phi = \Phi_{\text{ref}} + \Phi_1^e, \end{aligned} \tag{3}$$

where the subscript $_{\text{ref}}$ referred to the reference configuration and the subscript $_1$ to the perturbed state. The velocity $\vec{v}$ and the potential V are perturbations. In addition, we remain in the frame of a linear theory in term of $\vec{u}$. Finally, we express the mass conservation from the displacement field. We thus obtain the following equations:

$$\begin{aligned} &\rho_1^e + \vec{\nabla}\cdot(\rho_{\text{ref}}\vec{u}) = 0, \\ &\rho_{\text{ref}}\frac{\mathrm{d}^2\vec{u}}{\mathrm{dt}^2} = \vec{\nabla}\cdot(\overline{\overline{\Sigma_{\text{ref}}}} + \overline{\overline{\Sigma_1^e}}) + (\rho_{\text{ref}} + \rho_1^e)\vec{\nabla}(\Phi_{\text{ref}} + \Psi + \Phi_1^e) + \rho_{\text{ref}}\vec{\nabla}V, \\ &\Delta(\Phi_{\text{ref}} + \Phi_1^e) = -4\pi G(\rho_{\text{ref}} + \rho_1^e). \end{aligned} \tag{4}$$

We have to describe the different states appearing in these last equations, that is to say, the hydrostatic reference state and the elasto-gravitational perturbed state.

Let us first to recall the solutions for our elliptical hydrostatic reference model.

2.1. HYDROSTATIC REFERENCE MODEL

We are interested in the spheroidal Earth model which results from the uniform angular velocity Ω of the planet remaining in a state of hydrostatic equilibrium.

In a state without deformation, called reference state, the planet is elliptical and an internal particle is characterized by a density $\rho_{\text{ref}} = \rho_o + \delta\rho_o$, a stress $\overline{\overline{\Sigma_{\text{ref}}}} = \overline{\overline{\Sigma_o}} + \delta\overline{\overline{\Sigma_o}}$ which is assumed isotropic $[\overline{\overline{\Sigma_o}} + \delta\overline{\overline{\Sigma_o}} = -(P_o + \delta P_o)\overline{\overline{I}}]$, a gravitational potential $\Phi_{\text{ref}} = \Phi_o + \delta\Phi_o$ and the centrifugal potential Ψ. Potential, pressure and density are the sum of a radial term and a zonal degree 2 term. We note P_o, ρ_o, and Φ_o, respectively, the radial part of the pressure, density and gravitational potential induced by the radial part of the centrifugal potential Ψ_o; and $\delta P_o, \delta\rho_o$, and $\delta\Phi_o$, respectively, the zonal degree 2 perturbations of the spherical pressure, density and gravitational potential induced by the zonal part of the centrifugal potential. $P_2^0 = \frac{3\cos^2\theta - 1}{2}$ is the zonal degree two Legendre polynomial.

The mechanical and gravitational equilibrium equations governing the state of our reference Earth are:

$$\vec{\nabla}(P_o + \delta P_o) = (\rho_o + \delta\rho_o)\vec{\nabla}(\Phi_o + \delta\Phi_o + \Psi), \qquad (5)$$
$$\Delta(\Phi_o + \delta\Phi_o) = -4\pi G(\rho_o + \delta\rho_o).$$

We may solve first the radial part of the equation and then the zonal degree 2 part.

For our simple homogeneous incompressible model, there is no density perturbation $\delta\rho_o = 0$. Both the potential and the pressure are the sum of a radial term and a zonal degree two term. We note:

$$\Phi_o + \Psi + \delta\Phi_o = \tilde{\Phi}_o(r) + \delta\tilde{\Phi}_o(r, \theta, \varphi) \qquad (6)$$

with $\tilde{\Phi}_o(r) = \Phi_o(r) + \Psi_o(r)$ and $\delta\tilde{\Phi}_o(r, \theta, \varphi) = \delta\Phi_o(r, \theta, \varphi) + \Psi_o P_2^0(\theta)$, i.e., taking into account both the mass redistribution potential and the direct effect of the zonal degree 2 part of the centrifugal potential.

2.1.1. *Spherical reference model*

The radial part of the gravitational potential and hydrostatic pressure are governed by the following equations:

$$-\vec{\nabla} P_o + \rho_o \vec{\nabla}(\Phi_o + \Psi_o) = \vec{0} \tag{7}$$

and the Poisson equation:

$$\Delta \Phi_o = -4\pi G \rho_o. \tag{8}$$

For realistic Earth's model, we usually assume that the solutions of these equations are known from seismology.

For our incompressible homogeneous model, the solutions are:

$$\begin{aligned} \tilde{\Phi}_o(r) &= \Phi_o(r) + \Psi_o(r) = \frac{g_o a}{2}\left[3 - \frac{r^2}{a^2}\right] + \frac{\Omega^2 r^2}{3} \quad \text{and} \\ P_o(r) &= \frac{\rho_o g_o a}{2}\left[1 - \frac{r^2}{a^2}\right] - \frac{\rho_o \Omega^2 a^2}{3}\left[1 - \frac{r^2}{a^2}\right], \end{aligned} \tag{9}$$

where a is the Earth's radius and if we note M the mass of the planet, $g_o = \frac{GM}{a^2}$ the surface gravity.

2.1.2. *Elliptical Reference Model*

Subtracting Equations (7) and (8) from Equations (5), we rediscover that the zonal degree 2 fluid deformations obey to the Clairaut equation (e.g., Jeffreys, 1970), that-is-to-say depend only on the density stratification and on the angular velocity of the planet Ω. We note $\alpha(r)$ the hydrostatic flattening within the Earth. We have

$$\begin{aligned} \delta\tilde{\Phi}_o(r,\theta,\varphi) &= -\tfrac{2}{3}\alpha(r) g(r) r\, P_2^0, \\ \delta P_o(r,\theta,\varphi) &= -\tfrac{2}{3}\alpha(r)\rho_o(r) g(r) r\, P_2^0. \end{aligned} \tag{10}$$

There is a surface degree 2 topography, noted δd, such as: $\delta d = -\frac{2}{3}\alpha(a) a P_2^0$.

For our incompressible homogeneous model, we have $\alpha(r) = \frac{5}{4} q_o$ with $q_o = \frac{\Omega^2 a}{g_o}$ the geodetical constant.

2.1.3. *Remarks*

The radial terms $\tilde{\Phi}_o(r)$, $P_o(r)$ and also $\rho_o(r)$ are related to the theory of the reference state that-is-to-say to the theory of the interior of the planets. It is a static theory which takes into account most of the fundamental thermodynamical equations, of potentials, of phase changes and of states equations. It takes also into account the Poisson equation. This theory of the reference state is, for the Earth, known from seismology. It allows to have a mean model for the radially stratified Earth. The most classical is the PREM model derived by Dziewonski and Anderson (1981).

Note that Love (1911), in his analytical calculation, worked with respect to a non-rotating spherical reference Earth characterized by $\Phi_o(r)$, $P_o(r)$, and $\rho_o(r)$, that is to say, without taking into account the radial part of the centrifugal potential; he kept simultaneously, in the spheroidal perturbed state, the radial and the degree 2 term in both the incompressible pressure and the centrifugal potential.

In our approach, we choose to work with respect to a rotating spherical reference Earth which takes into account both the radial part of the centrifugal potential and the induced radial deformation, that-is-to-say a planet characterized by $\tilde{\Phi}_o(r)$, $P_o(r)$, and $\rho_o(r)$. This will change significantly the Love numbers perturbations and it will be taken into account when we will recombine the different perturbed states to build the final solution on the deformed surface of the planet.

Now, we will solve the elasto-gravitational theory for a spheroidal hydrostatic planet in two steps: first, we will compute the deformations $\vec{u}$ and Φ_1^e with respect to a spherical reference Earth's model, and then the perturbations of these deformations $\partial\vec{u}$ and $\partial\Phi_1^e$ due to the elliptical shape of the reference model (for a review of this perturbation theory, see Dahlen and Tromp, 1998).

Let us first recall the solutions for a spherical Earth.

2.2. ELASTOGRAVITATIONAL THEORY FOR A SPHERICAL EARTH

The equations governing the hydrostatic equilibrium of the spherical reference configuration are given by Equations (7–9). Taking into account these equations in the set of the elasto-gravitational equations (4), we have:

$$\begin{aligned}
&\rho_1^e + \vec{\nabla}\cdot(\rho_o\vec{u}) = 0,\\
&\rho_o\frac{d^2\vec{u}}{dt^2} = \vec{\nabla}\cdot\overline{\overline{\Sigma_1^e}} + \rho_o\vec{\nabla}\Phi_1^e + \rho_1^e\vec{\nabla}(\Phi_o+\Psi_o) + \rho_o\vec{\nabla}V,\\
&\Delta\Phi_1^e = -4\pi G\rho_1^e.
\end{aligned} \tag{11}$$

To obtain the whole set of equations, we need to add the rheological law linking $\overline{\overline{\Sigma_1^e}}$ to ϵ_{ij} . This relation is the one of an elastic body, but we have to careful if the reference state is pre-stressed. We can show (Dahlen and Tromp, 1998) that if the pre-stress is a pressure (no deviatoric part), then the best parametrization of the perturbations of the stress tensor is the Lagrangian parametrization. We thus introduce the tensor $\overline{\overline{\Sigma_1^l}}$ and the rheological elastic law for an isotropic medium is, noting K the incompressibility and μ the rigidity:

$$\Sigma_{1ij}^l = \left(K - \tfrac{2}{3}\mu\right)[\vec{\nabla}\cdot\vec{u}]\delta_{ij} + 2\mu\epsilon_{ij} \tag{12}$$

with the classical relation between the Lagrangian and Eulerian perturbation:

$$\overline{\overline{\Sigma_1^l}} = \overline{\overline{\Sigma_1^e}} + \vec{u}\cdot\vec{\nabla}\overline{\overline{\Sigma_o}}. \tag{13}$$

The conservation of impulsion may be consequently written:

$$\begin{aligned}\rho_o\frac{d^2\vec{u}}{dt^2} &= \vec{\nabla}\cdot\left[\left(K - \tfrac{2}{3}\mu\right)[\nabla\cdot\vec{u}]\overline{\overline{I}} + 2\mu\overline{\overline{\epsilon}}\right] + \vec{\nabla}[\vec{u}\cdot\vec{\nabla}P_o]\\ &\quad + \rho_o\vec{\nabla}\Phi_1^e + \rho_1^e\vec{\nabla}(\Phi_o + \Psi_o) + \rho_o\vec{\nabla}V,\end{aligned} \tag{14}$$

K and μ are related to the reference model.

The solution of the obtained set of equations is complex but, because the sphericity of the reference model, it is judicious to use spherical coordinates and to expand the parameters on a basis of spherical functions. Let r, θ, and φ the spherical coordinates in a frame centered to the mass center of the reference model; r is the radius, θ the colatitude, and φ the longitude. We use spherical harmonics $Y_n^m(\theta,\varphi)$ (Heiskanen and Moritz, 1967):

- For the displacement $\vec{u}$:

  $$\vec{u}(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[y_{1n}Y_n^m\vec{e}_r + y_{3n}r\vec{\nabla}Y_n^m + y_{7n}\vec{\nabla}\wedge(\vec{r}Y_n^m)], \tag{15}$$

  $y_{1n}(r)$ is the radial part of $\vec{u}$, $y_{3n}(r)$ the spheroidal part and $y_{7n}(r)$ the toroidal part.
  These coefficients are the spectral components of $\vec{u}$.
- For the Lagrangian traction $\vec{T}^l = \vec{n}_o\cdot\overline{\overline{\Sigma_1^l}}$, where $\vec{n}_o$ is the outer normal to the spherical reference surface ($\vec{n}_o = \vec{e}_r$).

  $$\vec{T}^l = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} y_{2n}(r)Y_n^m\vec{e}_r + ry_{4n}(r)\vec{\nabla}Y_n^m + y_{8n}(r)\vec{\nabla}\wedge(\vec{r}Y_n^m).$$

- For the potential

  $$V + \Phi_1^e = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} y_{5n}(r)Y_n^m \quad \text{with} \quad V = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} V_n^m\frac{r^n}{a^n}Y_n^m.$$

- For the Lagrangian attraction, we introduce:

$$y_{6n} = \frac{\mathrm{d}y_{5n}}{\mathrm{d}r} - 4\pi G\rho_o y_{1n}(r).$$

Using these notations, the set of Equations (4) may be written as a differential system with 8 equations of first-order (the indicial n is omitted for simplicity). In the frequency (noted ω) domain, we have (Alterman et al., 1959), where the $\cdot$ denotes the radial derivative:

$$\begin{aligned}
\dot{y}_1 &= -\frac{2(K-\frac{2}{3}\mu)}{(K+\frac{4}{3}\mu)}\frac{y_1}{r} + \frac{1}{(K+\frac{4}{3}\mu)}y_2 + \frac{n(n+1)(K-\frac{2}{3}\mu)}{(K+\frac{4}{3}\mu)}\frac{y_3}{r},\\
\dot{y}_2 &= \left[-4\rho_o g + \frac{2}{3}\rho_o\Omega^2 r - \omega^2\rho_o r + \frac{12\mu K}{(K+\frac{4}{3}\mu)r}\right]\frac{y_1}{r} - \frac{4\mu}{(K+\frac{4}{3}\mu)}\frac{y_2}{r}\\
&\quad + n(n+1)\left[\rho_o g - \frac{6\mu K}{(K+\frac{4}{3}\mu)r}\right]\frac{y_3}{r} + \frac{n(n+1)y_4}{r} - \rho_o y_6,\\
\dot{y}_3 &= -\frac{y_1}{r} + \frac{y_3}{r} + \frac{y_4}{\mu},\\
\dot{y}_4 &= \left[\rho_o g - \frac{2}{3}\rho_o\Omega^2 r - \frac{2\mu(3K)}{(K+\frac{4}{3}\mu)r}\right]\frac{y_1}{r}\\
&\quad + \left\{-\omega^2\rho_o r + \frac{2\mu[(K-\frac{2}{3}\mu)(2n^2+2n-1)+2\mu(n^2+n-1)]}{(K+\frac{4}{3}\mu)r}\right\}\frac{y_3}{r}\\
&\quad - \frac{(K-\frac{2}{3}\mu)}{(K+\frac{4}{3}\mu)}\frac{y_2}{r} - \frac{3y_4}{r} - \rho_o\frac{y_5}{r},\\
\dot{y}_5 &= 4\pi G\rho_o y_1 + y_6,\\
\dot{y}_6 &= -4\pi G\rho_o n(n+1)\frac{y_3}{r} + \frac{n(n+1)}{r}\frac{y_5}{r} - \frac{2y_6}{r},\\
\dot{y}_7 &= \frac{y_7}{r} + \frac{y_8}{\mu},\\
\dot{y}_8 &= \left[-\omega^2\rho_o r + \frac{\mu(n^2+n-2)}{r}\right]\frac{y_7}{r} - \frac{3y_8}{r}
\end{aligned} \tag{16}$$

The density $\rho_o(r)$, the rigidity $\mu(r)$ and the incompressibility $K(r)$ depend on the radial stratification of the reference Earth's model. $g(r)$ is the gravity. For a homogeneous reference sphere, we have $g(r) = g_o\frac{r}{a}$, where g_o is the surface gravity. Note that in the second and fourth equation, there is a term related to the radial part of the centrifugal potential (i.e., proportional to Ω^2).

This system describes the elasto-gravitational behavior within the elastic parts of a planet. It permits also the study of the seismic modes. For the studies related to deformations with periods large in comparison with one hour, we classically assume that $\omega = 0$ in this system: we talk then of static deformations.

To solve the elasto-gravitational equations, we have to add boundary conditions:

- The displacement and the attraction have to vanish at the center of mass ($r = 0$).
- At each internal interface of the reference spherical model, the y_i are continuous.
- At the surface of the reference spherical Earth's model ($r = a$), we have:

$$\left[\frac{\partial \Phi_1^e}{\partial r} + 4\pi G\rho_o u_r\right]_{a-}^{a+} = 0 \quad \text{or} \quad y_6(a) + \frac{n+1}{a} y_5(a) = \frac{2n+1}{a} V_n. \tag{17}$$

- In absence of external pressure or traction acting on this Earth surface, the Lagrangian traction has to vanish at $r = a$:

$$\vec{T}^l(a) = \vec{0}. \tag{18}$$

For an incompressible homogeneous Earth model, the static solutions may be written:

$$\begin{aligned}
y_1(r) &= C_3 r^{n+1} + C_4 r^{n-1},\\
y_2(r) &= 2\mu\left[\frac{n^2 - n - 3}{n} C_3 r^n + (n-1) C_4 r^{n-2}\right]\\
&\quad + \frac{4}{3}\pi G\rho_o^2 \left[C_3 r^{n+2} + C_4 r^n\right]\left(1 - \frac{2}{3} q_o\right) - \rho_o C_5 r^n,\\
y_3(r) &= \frac{n+3}{n(n+1)} C_3 r^{n+1} + \frac{C_4}{n} r^{n-1},\\
y_4(r) &= 2\mu\left[\frac{n+2}{n+1} C_3 r^n + \frac{n-1}{n} C_4 r^{n-2}\right],\\
y_5(r) &= C_5 r^n,\\
y_6(r) &= n C_5 r^{n-1} - 4\pi G\rho_o [C_3 r^{n+1} + C_4 r^{n-1}].
\end{aligned} \tag{19}$$

Because of the static assumption (i.e., $\omega = 0$), the solutions only contain terms in positive powers of the radial coordinate instead of terms of spherical Bessel functions (Love, 1911). Note that in the y_2 propagator, there is a term in q_o the geodetical constant defined in Section 2.1.2, i.e., related to the radial part of the centrifugal potential.

The constants are determined from the boundary condition which may be written for a degree n volumic potential V_n:

$$\begin{aligned} &y_2(a)=0,\\ &y_4(a)=0,\\ &y_6(a)+\frac{n+1}{a}y_5(a)=\frac{2n+1}{a}V_n. \end{aligned} \tag{20}$$

For $n=1$, we have to modify these boundary conditions in order to take into account the conservation of the center of mass ($y_5(a)=0$). We do not take into account this particular case in this paper where we are interested in the degree 2 tidal potential.

We introduce $\delta X_n=\frac{q_o}{3}\frac{2n+1}{n-1}\frac{1}{1+\overline{\mu}}$, where q_o is the geodetical constant and $\bar{\mu}_n=\frac{2n^2+4n+3}{n}\frac{\mu}{\rho_o g_o a}$; the constants are:

$$\begin{aligned} &C_3=-\frac{1}{2}\frac{(n+1)}{\overline{\mu}_n+1}\frac{1}{a^n}\frac{V_n}{g_o a}[1+\delta X_n],\\ &C_4=\frac{n(n+2)}{2(n-1)(1+\overline{\mu})}\frac{1}{a^{n-2}}\frac{V_n}{g_o a}[1+\delta X_n],\\ &C_5=\left[1+\frac{3}{2(n-1)(1+\overline{\mu})}(1+\delta X_n)\right]\frac{1}{a^n}V_n. \end{aligned} \tag{21}$$

We introduce the classical tidal Love numbers (Love, 1911) defined by:

$$y_1(a)=h_n\frac{V_n}{g_o};\quad y_3(a)=l_n\frac{V_n}{g_o};\quad y_5(a)=(1+k_n)V_n.$$

For a homogeneous incompressible Earth model, these numbers may be analytically computed:

$$\begin{aligned} k_n&=\frac{3}{2(n-1)}\frac{1}{1+\bar{\mu}_n}[1+\delta X_n],\\ h_n&=\frac{2n+1}{2(n-1)}\frac{1}{1+\bar{\mu}_n}[1+\delta X_n],\\ l_n&=\frac{3}{2n(n-1)}\frac{1}{1+\overline{\mu}_n}[1+\delta X_n]. \end{aligned}$$

Taking $q_o=0$, we rediscover the classical Love solutions, which was computed for a non rotating spherical Earth.

For a degree 2 tidal potential we note, for simplicity, $\overline{\mu}_2=\overline{\mu}$ and we have:

$$k_2 = \frac{3}{2}\frac{1}{1+\bar{\mu}}\left[1+\frac{5}{3}\frac{q_o}{1+\bar{\mu}}\right]; \quad h_2 = \frac{5}{2}\frac{1}{1+\bar{\mu}}\left[1+\frac{5}{3}\frac{q_o}{1+\bar{\mu}}\right];$$
$$l_2 = \frac{3}{4}\frac{1}{1+\bar{\mu}}\left[1+\frac{5}{3}\frac{q_o}{1+\bar{\mu}}\right].$$

Taking $a = 6{,}371{,}000\,\mathrm{m}$, $\rho_o = 5520\,\mathrm{kg\,m^{-3}}$, $\mu = 0.115 \times 10^{12}$ Pa, and $q_o = 1/289.9$, we obtain:

$$k_2 = 0.360932; \quad h_2 = 0.601553; \quad l_2 = 0.180466.$$

Let V_{M_2} be the degree 2 tidal potential induced by the semi-diurnal lunar wave M_2:

$$V_{M_2} = V_o\, 3\sin^2\theta\, \cos(\sigma t + 2\varphi)\frac{r^2}{a^2}$$

We find the classical solutions for an incompressible homogeneous Earth model:

$$\begin{aligned}
u_r(r,\theta,\varphi) &= \frac{3}{2}\sin^2\theta\frac{V_o}{g_o}\frac{1}{1+\bar{\mu}}\cos(\sigma t+2\varphi)\left[-3\frac{r^3}{a^3}+8\frac{r}{a}\right](1+\delta X_2),\\
u_\theta(r,\theta,\varphi) &= \frac{3}{2}\sin\theta\cos\theta\frac{V_o}{g_o}\frac{1}{1+\bar{\mu}}\cos(\sigma t+2\varphi)\left[-5\frac{r^3}{a^3}+8\frac{r}{a}\right](1+\delta X_2),\\
u_\varphi(r,\theta,\varphi) &= -\frac{3}{2}\sin\theta\frac{V_o}{g_o}\frac{1}{1+\bar{\mu}}\sin(\sigma t+2\varphi)\left[-5\frac{r^3}{a^3}+8\frac{r}{a}\right](1+\delta X_2),\\
\Phi_1^e(r,\theta,\varphi) &= \frac{9}{2}\sin^2\theta V_o\frac{1}{1+\bar{\mu}}\cos(\sigma t+2\varphi)\frac{r^2}{a^2}(1+\delta X_2).
\end{aligned} \tag{22}$$

We plot, in Figure 1, the surface radial and tangential displacement, at $t=0$, induced by the tidal wave M_2, and the perturbed geoid $\frac{1}{g_o}[\Phi_1^e + V_{M_2}]_{r=a}$ in millimeter.

Neglecting the effects of the radial centrifugal potential ($q_o \simeq 0$), for simplicity, the Cauchy stress tensor becomes:

$$\begin{aligned}
\Sigma_{rr}^l &= 3\sin^2\theta\frac{\mu V_o}{g_o a(1+\bar{\mu})}\cos(\sigma t+2\varphi)\left[8-9\frac{r^2}{a^2}\right]-P_1,\\
\Sigma_{r\theta}^l &= 24\sin\theta\cos\theta\frac{\mu V_o}{g_o a(1+\bar{\mu})}\cos(\sigma t+2\varphi)\left[1-\frac{r^2}{a^2}\right],\\
\Sigma_{r\varphi}^l &= -24\sin\theta\frac{\mu V_o}{g_o a(1+\bar{\mu})}\sin(\sigma t+2\varphi)\left[1-\frac{r^2}{a^2}\right],\\
\Sigma_{\theta\theta}^l &= 3\frac{\mu V_o}{g_o a(1+\bar{\mu})}\cos(\sigma t+2\varphi)\left[8\cos^2\theta+\frac{r^2}{a^2}(2-7\cos^2\theta)\right]-P_1,\\
\Sigma_{\varphi\varphi}^l &= 3\frac{\mu V_o}{g_o a(1+\bar{\mu})}\cos(\sigma t+2\varphi)\left[-8+\frac{r^2}{a^2}(7-2\cos^2\theta)\right]-P_1,\\
\Sigma_{\theta\varphi}^l &= 3\frac{\mu V_o}{g_o a(1+\bar{\mu})}\sin(\sigma t+2\varphi)\cos\theta\left[5\frac{r^2}{a^2}-8\right],
\end{aligned}$$

where P_1 is a pressure appearing in this simple incompressible case and corresponding to $K\ \mathrm{div}\vec{u}$

$$P_1(r,\theta,\varphi) = -\frac{3}{2}\sin^2\theta\,\frac{\rho_o V_o}{1+\bar{\mu}}\cos(\sigma t + 2\varphi)\frac{r^2}{a^2}\left[-3\frac{r^2}{a^2} + 3 + \frac{4}{19}\bar{\mu}\right].$$

The order of magnitude of these stresses is about some thousand Pascal.

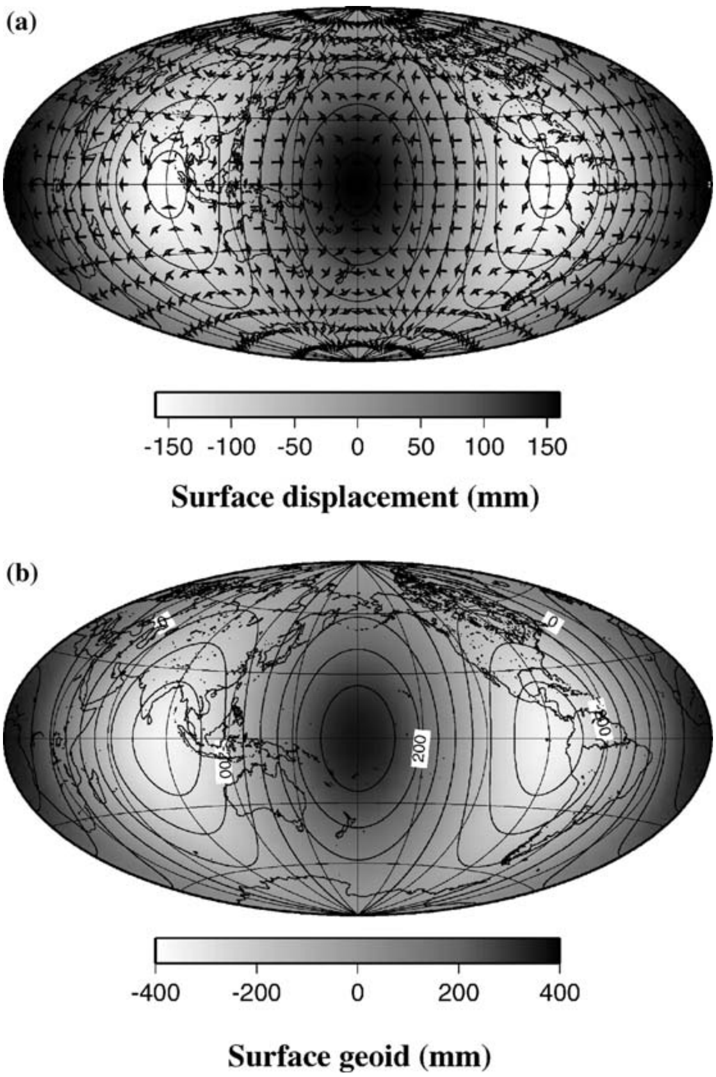

Figure 1. Spherical homogeneous incompressible Earth model with a radius a = 6371 km, a density $\rho_o = 5520\,\text{kg/m}^3$ and a rigidity $\mu = 0.115 \times 10^{12}$ Pa: (a) surface displacement in millimeter: the contour interval for the radial displacement is 50 mm, and the scale of the tangential displacement vector is 1 cm for 200 mm; (b) surface geoid in mm: the contour interval is 100 mm.

2.3. ELASTO-GRAVITATIONAL THEORY FOR A SPHEROIDAL HYDROSTATIC PLANET

In this part, we investigate the effects of the spheroidal shape of the Earth on the body tides. The reference ellipsoidal hydrostatic model is described as a perturbation of the spherical reference state [see Equation (5)].

The elasto-gravitational deformations, induced by a tidal potential V when the reference model is assumed to be ellipsoidal, may be written as the sum of the spherical elasto-gravitational deformations plus a perturbation.

$$\overline{\overline{\Sigma_1^l}} + \overline{\overline{\delta\Sigma_1^l}}, \quad \vec{u} + \vec{\delta u}, \quad \Phi_1^e + \delta\Phi_1^e.$$

Taking into account the equations governing the reference state (5), the equilibrium and Poisson equations become:

$$\begin{aligned}(\rho_o + \delta\rho_o)\frac{d^2(\vec{u} + \vec{\delta u})}{dt^2} &= \operatorname{div}\left(\overline{\overline{\Sigma_1^l}} + \delta\overline{\overline{\Sigma_1^l}} + [(\vec{u} + \vec{\delta u})\cdot\nabla](P_o + \delta P_o)\overline{\overline{I}}\right) \\ &\quad + (\rho_o + \delta\rho_o)\vec{\nabla}(\Phi_1^e + V + \delta\Phi_1^e) \\ &\quad - \operatorname{div}[(\rho_o + \delta\rho_o)(\vec{u} + \vec{\delta u})]\vec{\nabla}(\widetilde{\Phi}_o + \delta\widetilde{\Phi}_o), \\ \Delta[\Phi_1^e + \delta\Phi_1^e] &= 4\pi G \operatorname{div}[(\rho_o + \delta\rho_o)(\vec{u} + \vec{\delta u})]. \end{aligned} \tag{23}$$

Subtracting the equilibrium and Poisson equations for a spherical Earth (11) and neglecting the second order terms (such as $\delta\rho_o\vec{\delta u}\ldots$), we obtain:

$$\begin{aligned}\delta\rho_o\frac{d^2\vec{u}}{dt^2} + \rho_o\frac{d^2\vec{\delta u}}{dt^2} &= \operatorname{div}\left[\overline{\overline{\delta\Sigma_1^l}} + (\vec{u}\cdot\nabla)\delta P_o\overline{\overline{I}} + (\vec{\delta u}\cdot\nabla)P_o\overline{\overline{I}}\right] \\ &\quad + \delta\rho_o\vec{\nabla}(\Phi_1^e + V) + \rho_o\vec{\nabla}\delta\Phi_1^e \\ &\quad - \operatorname{div}[\rho_o\vec{\delta u} + \delta\rho_o\vec{u}]\vec{\nabla}\widetilde{\Phi}_o - \operatorname{div}(\rho_o\vec{u})\vec{\nabla}\delta\widetilde{\Phi}_o, \\ \Delta\delta\Phi_1^e &= 4\pi G \operatorname{div}[\rho_o\vec{\delta u} + \delta\rho_o\vec{u}]. \end{aligned} \tag{24}$$

The perturbed Cauchy stress tensor $\overline{\overline{\delta\Sigma_1^l}}$ may be written (for more details, see Dahlen and Tromp, 1998, p. 79):

$$(\delta\Sigma_1^l)_{ij} = \Gamma_{ijkl}(\delta u_l)_{,k} + \delta\Gamma_{ijkl}(u_l)_{,k} \tag{25}$$

with

$$\Gamma_{ijkl} = (K - \frac{2}{3}\mu)\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

and

$$\delta\Gamma_{ijkl} = (\delta K - \frac{2}{3}\delta\mu)\delta_{ij}\delta_{kl} + \delta\mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}],$$

$K(r)$ and $\mu(r)$ are respectively the radial part of the incompressibility and of the shear modulus, and δK and $\delta\mu$ the lateral variations.

Note that the set of Equation (24) may be written under the following form:

$$\rho_o \frac{d^2 \delta\vec{u}}{dt^2} - \mathrm{div}\left[\overline{\overline{\Gamma}} : (\vec{\nabla}\delta\vec{u}) + (\delta\vec{u}\cdot\nabla) P_o \overline{\overline{I}}\right] - \rho_o \vec{\nabla}\delta\Phi_1^e + \mathrm{div}(\rho_o \delta\vec{u})\vec{\nabla}\widetilde{\Phi}_o = \vec{F}_1,$$
$$\Delta\delta\Phi_1^e - 4\pi G \mathrm{div}(\rho_o \delta\vec{u}) = \vec{F}_2 \tag{26}$$

with

$$\vec{F}_1 = -\delta\rho_o \frac{d^2\vec{u}}{dt^2} + \mathrm{div}\left[\left(\overline{\overline{\delta\Gamma}}\right) : \left(\vec{\nabla}\vec{u}\right) + (\vec{u}\cdot\nabla)\delta P_o \overline{\overline{I}}\right] + \delta\rho_o \vec{\nabla}(\Phi_1^e + V)$$
$$-\mathrm{div}[\delta\rho_o \vec{u}]\vec{\nabla}\widetilde{\Phi}_o - \mathrm{div}(\rho_o \vec{u})\vec{\nabla}\delta\widetilde{\Phi}_o,$$
$$\vec{F}_2 = 4\pi G \mathrm{div}(\delta\rho_o \vec{u}).$$

The elasto-gravitational operator applied to $\delta\vec{u}$ and to $\delta\Phi_1^e$ is the same as the one applied to $\vec{u}$ and Φ_1^e in Equation (11) relative to a spherical Earth.

The boundary conditions are unchanged, that-is-to-say the displacement, the tractions, the potential and the gravity are continuous within the planet and at the Earth's deformed surface. Nevertheless, as they will be written at the interfaces of the reference radial sphere (in order to subtract the boundary conditions associated with $\vec{u}$ and Φ_1^e, it is necessary to take into account the topography at each interface (noted δd at the outer surface) and to introduce a non-radial normal $\vec{n}'_o = \vec{n}_o - \vec{\nabla}\vec{\nabla}^S\delta d$, where $\vec{\nabla}^S$ is the tangential gradient and $\vec{n}_o = \vec{e}_r$. Dahlen and Tromp (1998) showed that the boundary conditions at $r = a$ and at each solid–solid interface of the spherical reference model are:

- $\left[\delta\vec{u}\right]_-^+ = \left[-\delta d \partial_{\vec{n}_o}\vec{u}\right]_-^+,$
- $\left[\vec{n}_o \cdot \delta\overline{\overline{\Sigma_1^l}}\right]_-^+ = \left[-\delta d \vec{n}_o \cdot \partial_{\vec{n}_o}\overline{\overline{\delta\Sigma_1^l}} + \vec{\nabla}^S\delta d \cdot \overline{\overline{\Sigma_1^l}}\right]_-^+,$
- $\left[\delta\phi_1^e\right]_-^+ = \left[-\delta d\, \partial_{\vec{n}_o}\Phi_1^e\right]_-^+$
- $\left[\vec{n}_o \cdot \delta\vec{\xi}\right]_-^+ = \left[-\delta d\, \vec{n}_o \cdot \partial_{\vec{n}_o}\vec{\xi} + \vec{\nabla}^S\delta d \cdot \vec{\xi}\right]_-^+$

with

$$\vec{\xi} = -\vec{\nabla}\Phi_1^e + 4\pi G \rho_o \vec{u} \quad \text{and}$$
$$\delta\vec{\xi} = -\vec{\nabla}\delta\Phi_1^e + 4\pi G(\delta\rho_o \vec{u} + \rho_o \delta\vec{u}) \tag{27}$$

3. Analytical Solutions for an Elastic Incompressible Homogeneous Spheroidal Earth's Model

3.1. EQUATIONS AND PROPAGATORS

We intend to solve the Equation (26). The unknowns are $\vec{\delta u}$ and $\delta\Phi_1^e$. For a spheroidal homogeneous hydrostatic reference Earth, we have $\delta\rho_o = 0$, $\delta K = \delta\mu = 0$ and consequently $\vec{F_1}$ and $\vec{F_2}$ may be simply written:

$$\vec{F_1} = \text{div}\left[+(\vec{u}\cdot\nabla)(\delta P_o \overline{\overline{I}})\right] \qquad F_2 = 0. \tag{28}$$

We introduce a function $X(r,\theta,\varphi) = \left(u_r\partial_r + \frac{1}{r}u_\theta\partial_\theta + \frac{1}{r\sin\theta}u_\varphi\partial_\varphi\right)\delta P_o$. We expand X into spherical harmonics

$$X(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} x_n^m(r) Y_n^m(\theta,\varphi).$$

We note for the spherical functions: $Y_n^{mc}(\theta,\varphi) = P_n^m(\cos\theta)\cos m\varphi$ et $Y_n^{ms}(\theta,\varphi) = P_n^m(\cos\theta)\sin m\varphi$, where P_n^m are the non-normalized Legendre polynomials.

$$\begin{aligned} Y_2^{2c} &= 3\sin^2(\theta)\cos 2\varphi, & Y_4^{2c} &= \tfrac{15}{2}\sin^2(\theta)[7\cos^2(\theta)-1]\cos 2\varphi,\\ Y_2^{2s} &= 3\sin^2(\theta)\sin 2\varphi, & Y_4^{2s} &= \tfrac{15}{2}\sin^2(\theta)[7\cos^2(\theta)-1]\sin 2\varphi. \end{aligned} \tag{29}$$

We find:

$$X(r,\theta,\varphi) = x_2(r)\left[\cos\sigma t Y_2^{2c} - \sin\sigma t Y_2^{2s}\right] + x_4(r)\left[\cos\sigma t Y_4^{2c} - \sin\sigma t Y_4^{2s}\right]$$

with

$$\begin{aligned} x_2(r) &= -\frac{1}{21}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}\frac{r^2}{a^2}\left[27\frac{r^2}{a^2} - 56\right],\\ x_4(r) &= -\frac{4}{35}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}\frac{r^4}{a^4}. \end{aligned} \tag{30}$$

Similarly to the spherical case, we expand the perturbations in spherical harmonics:

- For the perturbation of the displacement:

$$\begin{aligned} \vec{\delta u} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} & \delta y_{1n}^m(r) Y_n^m(\theta,\varphi)\frac{\vec{r}}{r}\\ & + r\delta y_{3n}^m(r)\vec{\nabla} Y_n^m(\theta,\varphi) + \delta y_{7n}^m(r)\vec{\nabla}\wedge\vec{r}Y_n^m(\theta,\varphi). \end{aligned} \tag{31}$$

- For the perturbation of the Lagrangian tractions:

$$\vec{n}_o \cdot \overline{\overline{\delta\Sigma_1^l}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \delta y_{2n}^m(r) Y_n^m(\theta,\varphi)\frac{\vec{r}}{r} + r\delta y_{4n}^m(r)\vec{\nabla} Y_n^m(\theta,\varphi) + \delta y_{8n}^m(r)\vec{\nabla}\wedge\vec{r}Y_n^m(\theta,\varphi). \quad (32)$$

- For the perturbation of the potential:

$$\delta\Phi_1^e = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \delta y_{5n}^m(r) Y_n^m(\theta,\varphi). \quad (33)$$

- For the perturbation of the gravity:

$$\delta y_6 = \frac{d\delta y_5}{dr} - 4\pi G\rho_o \delta y_1(r). \quad (34)$$

The perturbed equations of the elasto-gravity may be written as a first-order differential system:

$$\frac{d}{dr}\delta y_i(r) = A_{ij}(r)\delta y_j(r) + f_i(r) \quad \text{pour} \quad (i,j) = 1.8, \quad (35)$$

where $A_{ij}(r)$ is the matrix of the spherical y_i system (16). It depends on the radial density, shear modulus and compressibility.

$f_i(r)$ is relative to the vectors $\vec{F}_1$ and $\vec{F}_2$. We have: $\vec{f} = [0, \partial_r x_n^m(r), 0, x_n^m(r)/r, 0, 0, 0, 0]$. For an incompressible homogeneous Earth, the solutions of the perturbed system (35) may be written, where we omit, for simplicity, the degree n and order m for each δy_i and each constant.

$$\begin{aligned}
\delta y_1 &= \delta C_3 r^{n+1} + \delta C_4 r^{n-1},\\
\delta y_2 &= 2\mu\left[\frac{n^2-n-3}{n}\delta C_3 r^n + (n-1)\delta C_4 r^{n-2}\right]\\
&\quad + \frac{\rho_o g_o}{a}\left[\delta C_3 r^{n+2} + \delta C_4 r^n\right] - \rho_o \delta C_5 r^n - x_n^m(r),\\
\delta y_3 &= \frac{n+3}{n(n+1)}\delta C_3 r^{n+1} + \frac{\delta C_4}{n} r^{n-1},\\
\delta y_4 &= 2\mu\left[\frac{n+2}{n+1}\delta C_3 r^n + \frac{n-1}{n}\delta C_4 r^{n-2}\right],\\
\delta y_5 &= \delta C_5 r^n,\\
\delta y_6 &= n\delta C_5 r^{n-1} - 4\pi G\rho_o\left[\delta C_3 r^{n+1} + \delta C_4 r^{n-1}\right],\\
\delta y_7 &= \delta C_7 r^n,\\
\delta y_8 &= \mu(n-1)\delta C_7 r^{(n-1)},
\end{aligned} \quad (36)$$

where the terms in $q_o \delta C_3$ and $q_o \delta C_4$ have been neglected because they are of second-order of our approximation.

Note that for our simple homogeneous incompressible model, because $\vec{F}_1$ is simply a gradient, the x_n^m term only appears in the solution δy_2 but vanishes in the solution δy_4.

The constants δC_3, δC_4, δC_5, and δC_7 have to be determined from the boundary conditions (27). The details of the computation of each degree n and order m constants are given in Appendix A.

3.2. SOLUTIONS

Knowing the constants, we know the perturbations of the displacement, of the gravitational potential and of the stress tensor within the entire planet.

At the surface of the reference Earth's model ($r = a$), we introduce perturbations of the Love numbers such as:

$$\begin{aligned} \delta\vec{u} = \frac{V_o}{g_o}\{&(\delta h_2 P_2^2 + \delta h_4 P_4^2)\cos(\sigma t + 2\varphi)\vec{e}_r \\ &+\vec{\nabla}[(\delta l_2 P_2^2 + \delta l_4 P_4^2)\cos(\sigma t + 2\varphi)] \\ &+\vec{e}_r \wedge \vec{\nabla}[\delta l_3 P_3^2 \sin(\sigma t + 2\varphi)]\}, \\ \delta\Phi_1^e = V_o[&(\delta k_2 P_2^2 + \delta k_4 P_4^2)\cos(\sigma t + 2\varphi)] \end{aligned} \tag{37}$$

with

$$\begin{aligned} &\delta h_2 = \tfrac{1}{399}\alpha\tfrac{(653\bar{\mu}+1349)}{(1+\bar{\mu})^2}, \quad \delta h_4 = -\tfrac{4}{35}\alpha\tfrac{(68\bar{\mu}+95)}{(1+\bar{\mu})(51\bar{\mu}+38)}, \\ &\delta l_2 = \tfrac{1}{3990}\alpha\tfrac{(3365\bar{\mu}+5453)}{(1+\bar{\mu})^2}, \quad \delta l_4 = -\tfrac{1}{315}\alpha\tfrac{(612\bar{\mu}+589)}{(1+\bar{\mu})(51\bar{\mu}+38)}, \quad \delta l_3 = \tfrac{1}{15}\alpha\tfrac{1}{(1+\bar{\mu})}, \\ &\delta k_2 = \tfrac{2}{655}\alpha\tfrac{(127\bar{\mu}+475)}{(1+\bar{\mu})^2}, \quad \delta k_4 = \tfrac{34}{15}\alpha\tfrac{\bar{\mu}}{(1+\bar{\mu})(51\bar{\mu}+38)}. \end{aligned} \tag{38}$$

For our homogeneous incompressible Earth with a radius $a = 6371\,\text{km}$, a density $\rho_o = 5520\,\text{kg/m}^3$, a rigidity $\mu = 0.115 \times 10^{12}\,\text{Pa}$ and an hydrostatic flattening $\alpha = \frac{5}{4}q_o = \frac{1}{232}$, we find:

$$\begin{aligned} &\delta h_2 = 0.002130, \quad \delta h_4 = -0.000184, \\ &\delta l_2 = 0.001004, \quad \delta l_4 = -0.000042, \quad \delta l_3 = 0.000069, \\ &\delta k_2 = 0.000656, \quad \delta k_4 = 0.000037. \end{aligned} \tag{39}$$

The perturbation is of about 10^{-3} for the degree 2, and about 10^{-4} for the degree 4. The order of magnitude of the radial and tangential component of the perturbation $\delta\vec{u}$ of the displacement as well as the perturbed potential $\delta\Phi_1^e(a)/g_o$ is the millimeter and is consequently significant, because it is well known that for space geodesy, it is now necessary to achieve the mm level in the tidal displacements.

3.3. COMPARISON WITH THE LOVE'S SOLUTIONS

Our results for δh_4 and δk_4 are identical to those obtained by Love (1911) and corrected by Wang (1994). On the contrary our results for the degree 2 perturbations differ from the Love solutions. This is because Love started with a reference model which did not take into account the radial part of the centrifugal potential (and the associated radial part of the pressure given in Equation (9)). Consequently in his δP_o he had a radial term:

$$\delta P_o = -\frac{2}{3}\alpha\, \rho_o g_o a\, \frac{r^2}{a^2} P_2^0 - \rho_o g_o a\, \frac{q_o}{3}\left(1 - \frac{r^2}{a^2}\right).$$

In our approach, this term only changes the right-hand side of Equation (26) and thus appears in the $X(r,\theta,\varphi)$ function. If we use the Love's hypothesis, we will obtain, where the superscript ^{L} is related to the Love's solutions:

$$x_2^L(r) = -\frac{1}{35}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}\frac{r^2}{a^2}\left[73\frac{r^2}{a^2} - 168\right] \tag{40}$$

the x_4 coefficient is unchanged. The right-hand side of the boundary conditions in δy_2 (Equation (A14) in Appendix A) will also be changed:

$$\begin{aligned}
&-\rho_o a^2 \left(\delta C_5^c\right)_2^2 + \frac{\rho_o g_o a}{19}\left[(19-2\bar{\mu})a^2\left(\delta C_3^c\right)_2^2 + (19+4\bar{\mu})\left(\delta C_4^c\right)_2^2\right] \\
&\quad = \frac{1}{399}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}(128\bar{\mu}+1311)\cos(\sigma t), \\
&-\rho_o a^2 \left(\delta C_5^s\right)_2^2 + \frac{\rho_o g_o a}{19}\left[(19-2\bar{\mu})a^2\left(\delta C_3^s\right)_2^2 + (19+4\bar{\mu})\left(\delta C_4^s\right)_2^2\right] \\
&\quad = -\frac{1}{399}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}(128\bar{\mu}+1311)\sin(\sigma t).
\end{aligned} \tag{41}$$

This will change the constants and finally the perturbations of the Love numbers will be:

$$\begin{aligned}
\delta h_2^L &= \frac{1}{399}\alpha\frac{(653\bar{\mu}+2679)}{(1+\bar{\mu})^2} = 0.002953, \\
\delta l_2^L &= \frac{1}{3990}\alpha\frac{(3365\bar{\mu}+9443)}{(1+\bar{\mu})^2} = 0.001251, \\
\delta k_2^L &= \frac{2}{665}\alpha\frac{(127\bar{\mu}+1140)}{(1+\bar{\mu})^2} = 0.001151.
\end{aligned} \tag{42}$$

Note that the discrepancy between δk_2^L and δk_2 is significant (about a factor 2). Consequently, it will be very important when we compare the Love numbers perturbations coming from different studies to be careful on the initial hypothesis concerning the reference model. This remark is also valid when we combine the different states of perturbations of our approach in order to compute some surface quantities, such as the gravity, on the deformed surface.

It is easy to relate our perturbed Love numbers to the ones computed by Love from:

$$\begin{aligned} \delta k_2^L &= \delta k_2 + \frac{3}{2}\frac{1}{1+\bar{\mu}}\delta X_2, \\ \delta h_2^L &= \delta h_2 + \frac{5}{2}\frac{1}{1+\bar{\mu}}\delta X_2, \\ \delta l_2^L &= \delta l_2 + \frac{3}{4}\frac{1}{1+\bar{\mu}}\delta X_2. \end{aligned} \tag{43}$$

4. Geodetic and Gravimetric Responses of the Earth to the Tides on the Spheroidal Deformed Surface

We intend to extend these results to some geodetic responses of the Earth to tidal forces such as the geoid and the topography on the deformed surface, the perturbation of the surface gravity field (with the associated gravimetric factor), the tilt and the deviation of the vertical with reference to the Earth's axis (for a review of these measurements of the Earth's tides, see Melchior, 1966).

The difficulty of defining the Love numbers or the tidal gravimetric factor for an ellipsoidal Earth has been pointed out because it is no more possible to define the Earth's response with only one parameter as it was the case for a spherical Earth. It is particularly the case for tilt for which there must be different Love numbers in different directions.

4.1. OUTWARD NORMALS

Let $\vec{n}_t$ be the outward normal of the surface topography and $\vec{n}_g$ the outward normal to the geoid. In our perturbation theory, the terms in $\delta\tilde{\Phi}_o\Phi_1^e$ have the same order of magnitude that terms in $\delta\Phi_1^e$. Consequently, to compute the normal of a given surface, we have to keep non linear terms. If we note $r(\theta,\varphi)=a[1+x(\theta,\varphi)]$ with $x(\theta,\varphi)\ll 1$, a given surface, the outer normal will be, in the second-order of approximation:

$$\begin{aligned} \vec{n} = \vec{e}_r &\left[1 - \frac{1}{2}(\partial_\theta x(\theta,\varphi))^2 - \frac{1}{2\sin^2\theta}\left(\partial_\varphi x(\theta,\varphi)\right)^2\right] \\ &- \partial_\theta x(\theta,\varphi)[1-x(\theta,\varphi)]\vec{e}_\theta - \frac{1}{\sin\theta}\partial_\varphi x(\theta,\varphi)[1-x(\theta,\varphi)]\vec{e}_\varphi. \end{aligned} \tag{44}$$

For the geoid, we have:

$$x(\theta,\varphi)=\left[\frac{\delta\tilde{\Phi}_o+\Phi_1^e+V_2+\delta\Phi_1^e+\delta d\,\partial_r(\Phi_1^e+V_2)}{g_o a}\right]_{r=a}$$

and consequently, for the normal:

$$\begin{aligned}\vec{n}_g=\vec{e}_r&\left[1-\vec{\nabla}^S\frac{\Phi_1^e+V_2}{g_o}\cdot\vec{\nabla}^S\frac{\delta\Phi_o}{g_o}\right]_{r=a}\\&-\frac{1}{g_o}\vec{\nabla}^S[\delta\tilde{\Phi}_o+\Phi_1^e+V_2+\delta\Phi_1^e+\delta d[\partial_r(\Phi_1^e+V_2)]-(\Phi_1^e+V_2)\frac{\delta\tilde{\Phi}_o}{g_o a}]_{r=a}\end{aligned} \quad (45)$$

For the surface topography, we have:

$$x(\theta,\varphi)=\left[\frac{\delta d+u_r+\delta u_r+\delta d\,\partial_r u_r}{a}\right]_{r=a}$$

and consequently, for the normal:

$$\vec{n}_t=\vec{e}_r[1-\vec{\nabla}^S u_r\cdot\vec{\nabla}^S\delta d]_{r=a}-\vec{\nabla}^S\left[\delta d+u_r+\delta u_r+\delta d[\partial_r u_r]-u_r\frac{\delta d}{a}\right]_{r=a}. \quad (46)$$

4.2. SURFACE DISPLACEMENT AND GEOID ON THE DEFORMED ELLIPSOIDAL OUTER SURFACE

We can define Love numbers related to the vertical surface displacement and to the tangential displacement on the ellipsoid. We first introduce a basis of vectors related to the ellipsoid:

$$\begin{aligned}\vec{E}_n&=\vec{e}_r-\vec{\nabla}^S\delta d,\\ \vec{E}_S&=\vec{e}_\theta+\partial_\theta\frac{\delta d}{a}\vec{e}_r,\\ \vec{E}_E&=\vec{e}_\varphi+\frac{1}{\sin\theta}\partial_\varphi\frac{\delta d}{a}\vec{e}_r.\end{aligned} \quad (47)$$

These vectors are normalized and orthogonal (in the first-order in $\frac{\delta d}{a}$). Note that for a axi-symmetric ellipsoid, $\partial_\varphi\frac{\delta d}{a}=0$ and consequently $\vec{E}_E=\vec{e}_\varphi$ is the east direction.

At the order of our approximation, the vertical displacement may be written:

$$u_n = \left[\vec{e}_r - \vec{\nabla}^S \delta d\right] \cdot \left[\vec{u} + \vec{\delta} u + (\delta d \vec{e}_r \cdot \vec{\nabla})\vec{u}\right]$$
$$= \frac{V_o}{g_o}\left[(h_2 + \Delta h_o) P_2^2 + \Delta h_+ P_4^2\right] \cos(\sigma t + 2\varphi) \quad (48)$$

with the Love numbers:

$$h_2 + \Delta h_o = h_2 + \delta h_2 - \frac{11}{21}\alpha \frac{1}{1+\overline{\mu}} = 0.603140$$
$$\triangle h_+ = \delta h_4 - \frac{1}{7}\alpha \frac{1}{1+\overline{\mu}} = -0.000332.$$

The tangential displacement on the ellipsoid is:

$$\vec{u}_H = \vec{u} + \vec{\delta u} + \delta d \partial_r \vec{u} - u_n \vec{E}_n = u_{HS} \vec{E}_S + u_{HE} \vec{E}_E \quad (49)$$

with

$$u_{HS} = u_\theta + \delta u_\theta + \delta d \partial_r u_\theta + \partial_\theta \frac{\delta d}{a} u_r, \quad (50)$$
$$u_{HE} = u_\varphi + \delta u_\varphi + \delta d \partial_r u_\varphi + \frac{1}{\sin\theta} \partial_\varphi \frac{\delta d}{a} u_r.$$

We introduce tangential Love numbers such as $\vec{u}_H$ may be written in the ellipsoidal vectors basis:

$$\vec{u}_H = \frac{V_o}{g_o} \left\{ \vec{\nabla}^S \left[\left((l_2 + \Delta l_o) P_2^2 + \Delta l_+ P_4^2\right) \cos(\sigma t + 2\varphi)\right] \right.$$
$$\left. + \vec{e}_r \wedge \vec{\nabla}^S \left[\triangle l_* P_3^2 \sin(\sigma t + 2\varphi)\right] \right\} \quad (51)$$

with

$$l_2 + \Delta l_o = l_2 + \delta l_2 + \frac{2}{21}\alpha l_2 = l_2 + \frac{1}{1995}\alpha \frac{(1825\bar{\mu} + 2869)}{(1+\bar{\mu})^2} = 0.181544,$$
$$\triangle l_+ = \delta l_4 - \frac{1}{35}\alpha l_2 = -\frac{1}{1260}\alpha \frac{(3825\bar{\mu} + 3382)}{(1+\bar{\mu})(51\bar{\mu} + 38)} = -0.000064, \quad (52)$$
$$\triangle l_* = \delta l_3 - \frac{1}{15}\alpha(2h_2 - l_2) = -\frac{13}{60}\alpha \frac{1}{(1+\bar{\mu})} = -0.000224.$$

We plot in Figure 2a the perturbation of vertical surface displacement (i.e. $u_n - u_r(a)$ and of the tangential surface displacement (i.e., $[u_{HS} - u_\theta]\vec{E}_S + [u_{HE} - u_\varphi]\vec{E}_E$), in millimeter.

Similarly, we introduce Love numbers related to the geoid on the deformed ellipsoidal outer surface:

$$
\begin{aligned}
&\frac{1}{g_o}\left[\Phi_1^e + V_2 + \delta\Phi_1^e + \delta d\,\partial_r(\Phi_1^e + V_2)\right]_{r=a} \\
&\quad = \frac{V_o}{g_o}\left[(1 + k_2 + \Delta k_o)P_2^2 + \Delta k_+ P_4^2\right]\cos(\sigma t + 2\varphi)
\end{aligned}
\tag{53}
$$

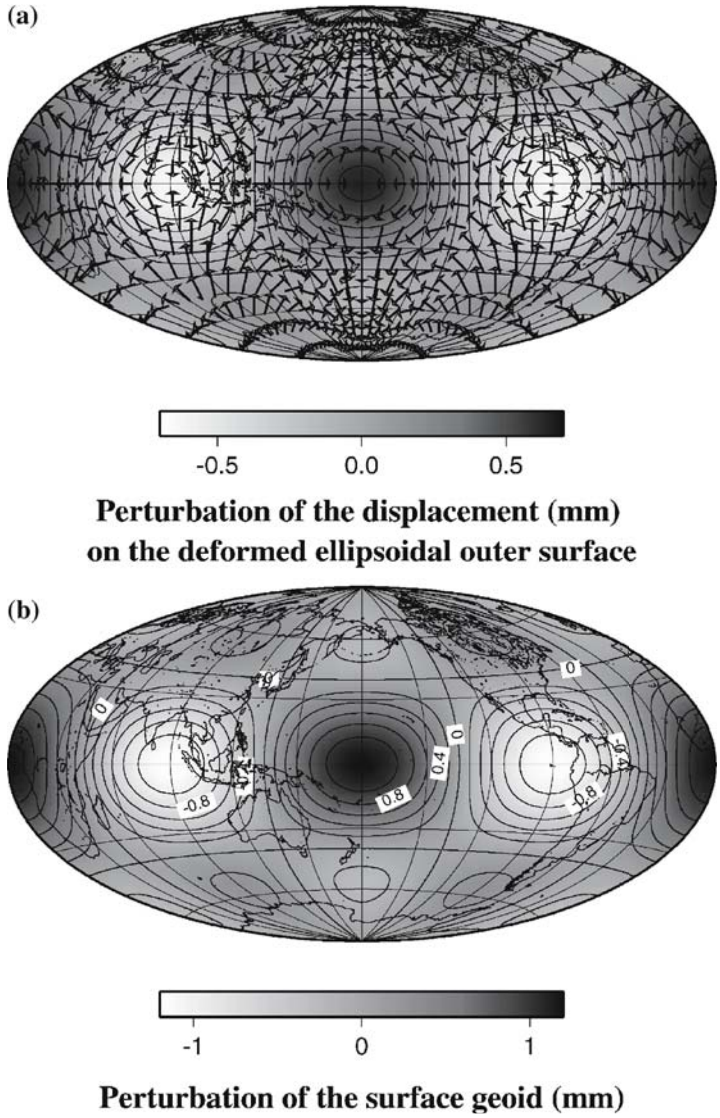

Figure 2. (a) Perturbation of the displacement on the deformed outer ellipsoidal surface: the contour interval for the vertical displacement $u_n - u_r(a)$ is 0.1 mm, and the scale for the tangential vector is 1 cm for 1 mm. (b) Perturbation of the geoid on the deformed outer ellipsoidal surface: the contour interval is 0.2 mm.

with

$$k_2+\Delta k_{\text{o}}=k_2+\delta k_2+\frac{8}{21}\alpha(1+k_2)=0.363824,$$
$$\triangle k_+=\delta k_4-\frac{4}{35}\alpha(1+k_2)=-0.000633.$$

We plot in Figure 2b the perturbed geoid on the outer surface $\frac{1}{g_{\text{o}}}\left[\delta\Phi_1^e+\delta d\,\partial_r(\Phi_1^e+V_2)\right]_{r=a}$, in millimeter.

4.3. SURFACE GRAVITY PERTURBATION

In this part, we are interested in the variations of the intensity of gravity which can be carried out with the help of gravimeters.

The attraction on the ellipsoidal deformed outer surface is:

$$\vec{A}=\vec{\nabla}\left[\tilde{\Phi}_{\text{o}}+\delta\tilde{\Phi}_{\text{o}}+\Phi_1^e+V_2+\delta\Phi_1^e\right]_{r=a} + \left[(\delta d\vec{e}_r+\vec{u}+\delta d\,\partial_r u_r\vec{e}_r+\vec{\delta u})\cdot\vec{\nabla}\right]\vec{\nabla}\left[\tilde{\Phi}_{\text{o}}+\delta\tilde{\Phi}_{\text{o}}+\Phi_1^e+V_2+\delta\Phi_1^e\right]_{r=a}. \tag{54}$$

In our order of approximation, we have:

$$\vec{A}=\vec{\nabla}\left(\tilde{\Phi}_{\text{o}}+\delta\tilde{\Phi}_{\text{o}}\right)+\delta d\,\partial_r\vec{\nabla}(\tilde{\Phi}_{\text{o}})+\vec{\nabla}(\Phi_1^e+V_2)+(\vec{u}\cdot\vec{\nabla})\vec{\nabla}\tilde{\Phi}_{\text{o}}+\vec{\nabla}\delta\Phi_1^e +(\vec{\delta u}\cdot\vec{\nabla})\vec{\nabla}\tilde{\Phi}_{\text{o}}+(\vec{u}\cdot\vec{\nabla})\vec{\nabla}\delta\tilde{\Phi}_{\text{o}}+\delta d\,\partial_r\vec{\nabla}(\Phi_1^e+V_2)+\partial_r u_r\delta d\,\partial_r\vec{\nabla}\tilde{\Phi}_{\text{o}}. \tag{55}$$

The gravity along the vertical at the station is $g=\vec{n}_{\text{g}}.\vec{A}$. We note $g=g_{\text{o}}+\delta g_{\text{o}}+g_1+\delta g_1$, with

$$g_{\text{o}}+\delta g_{\text{o}}=\partial_r\left(\tilde{\Phi}_{\text{o}}+\delta\tilde{\Phi}_{\text{o}}\right)+\delta d\,\partial_r^2\tilde{\Phi}_{\text{o}},$$
$$g_1=\partial_r\left(\Phi_1^e+V_2\right)+\left(\vec{u}\cdot\vec{\nabla}\right)\partial_r\tilde{\Phi}_{\text{o}},$$
$$\delta g_1=\partial_r\delta\Phi_1^e+\left(\vec{\delta u}\cdot\vec{\nabla}\right)\partial_r\tilde{\Phi}_{\text{o}}+\left(\vec{u}\cdot\vec{\nabla}\right)\partial_r\delta\tilde{\Phi}_{\text{o}}+\delta d\,\partial_r^2\left(\Phi_1^e+V_2\right) +\partial_r u_r\delta d\,\partial_r^2\tilde{\Phi}_{\text{o}}-\vec{\nabla}^S\left(\Phi_1^e+V_2\right)\cdot\vec{\nabla}^S\delta d. \tag{56}$$

We plot in Figure 3 the perturbation of the gravity induced by the M_2 tidal wave on the ellipsoidal deformed outer surface: g_1 in Figure 3a and δg_1 in Figure 3b. Note that there is a significant degree 4 order 2 component with an amplitude of about 200 nanogals. This perturbation should be detectable with the use of very accurate superconducting gravimeters when the oceanic effects are correctly modeled.

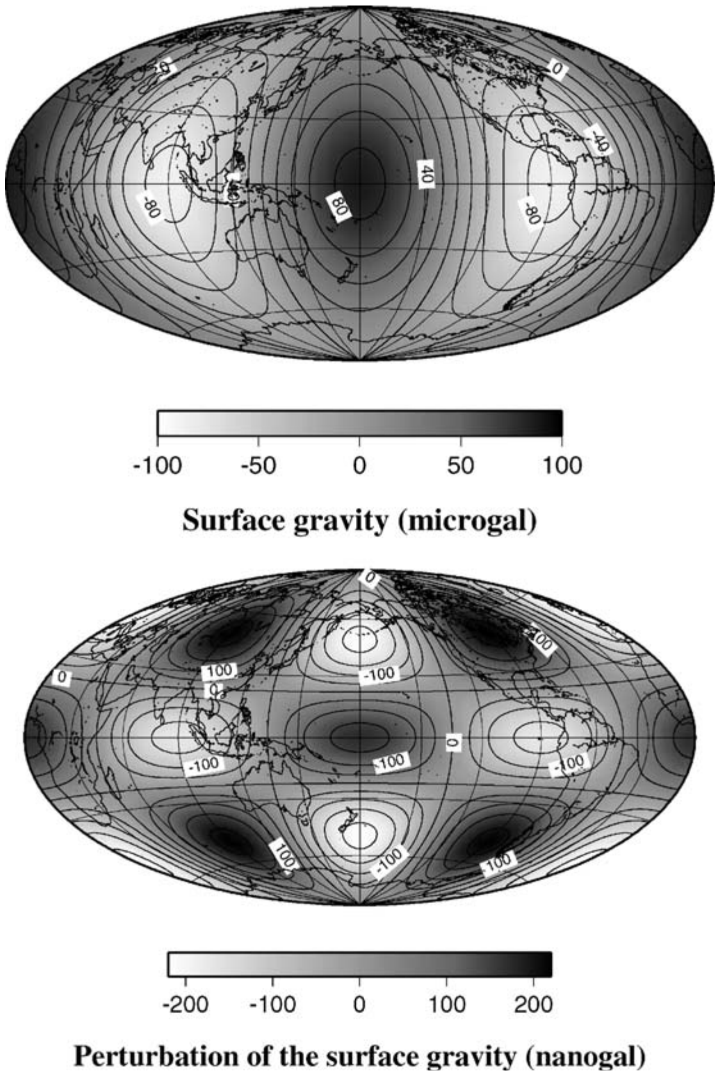

Figure 3. Perturbation of the gravity on the deformed outer ellipsoidal surface: g_1 on the top and δg_1 on the bottom.

Using the definition proposed by Wahr (1981) we define the gravimetric factor as the transfer function between the body tide signal measured along the vertical at the station by a gravimeter and the amplitude of the radial derivative of the tidal potential. We introduce the classical degree 2 gravimetric factor δ_2 such as:

$$g_1 = \frac{2V_o}{a}\delta_2 P_2^2 \cos(\sigma t + 2\varphi). \tag{57}$$

Because $\tilde{\Phi}_o$ is the sum of the spherical gravitational potential and of the radial part of the centrifugal potential, we have:

$$\partial_r^2(\tilde{\Phi}_o) = \frac{2g(r)}{r} - 4\pi G\rho_o + \frac{2}{3}\Omega^2 \tag{58}$$

and consequently, we have

$$\delta_2 = 1 + h_2\left(1 + \frac{m}{3}\right) - \frac{3}{2}k_2 = 1 + \frac{1}{4}\frac{1}{(1+\bar{\mu})} + \frac{\alpha}{3}\frac{3+2\bar{\mu}}{(1+\bar{\mu})^2}. \tag{59}$$

We define perturbations of this gravimetric factor such as:

$$\delta g_1(\theta, \varphi) = \frac{2V_o}{a}\cos(\sigma t + 2\varphi)\left[\delta\delta_2 P_2^2 + \delta\delta_4 P_4^2\right]$$

with, for our homogeneous incompressible Earth model:

$$\delta\delta_2 = \frac{\alpha}{3990}\frac{1805 + 729\bar{\mu} - 380\bar{\mu}^2}{(1+\bar{\mu})^2}, \quad \delta\delta_4 = -\frac{\alpha}{210}\frac{2166 + 4037\bar{\mu} + 1836\bar{\mu}^2}{(1+\bar{\mu})(51\bar{\mu} + 38)}.$$

Our result for $\delta\delta_4$ is identical to the one obtained from Equation (9) of Wang (1994) from the Eulerian potential in free space. For $\delta\delta_2$, there are discrepancies due to the difference in the initial spherical reference model (with or without the radial pressure and centrifugal potential). But our total gravimetric factor($\delta_2 + \delta\delta_2$) is equal to the one obtained using Equation (8) of Wang (1994) paper:

$$\delta_2 + \delta\delta_2 = 1 + \frac{1}{4}\frac{1}{(1+\bar{\mu})} + \frac{\alpha}{3990}\frac{5795 + 3389\bar{\mu} - 380\bar{\mu}^2}{(1+\bar{\mu})^2}$$

With the numerical values given in Section 3.2, we obtain:

$$\delta_2 + \delta\delta_2 = 1.060175, \quad \delta\delta_4 = -0.000824.$$

Note that these coefficients are related to non-normalized Legendre polynomials.

If the planet is rigid, our gravimetric factors will become:

$$\delta_2 = 1, \quad \delta\delta_2 = -\frac{2}{21}\alpha = -0.0004105, \quad \delta\delta_4 = -\frac{6}{35}\alpha = -0.0007389.$$

Note that the gravimetric factor is sometimes defined as the transfer function between the body tide signal measured along the vertical at the station by a gravimeter and the amplitude of the gradient of the external tidal potential along the perpendicular to the reference ellipsoid (e.g., Dehant et al., 1999).

We can also compute the components of the attraction in the basis of vectors related to the ellipsoid (defined in 47). We obtain, for the normal component: $\vec{A}\cdot\vec{E}_n = g$, where g is the gravity defined in Equation (56). For the horizontal component (i.e., the deviation of the vertical), noting $A_S = \vec{A}\cdot\vec{E}_S$ and $A_E = \vec{A}\cdot\vec{E}_E$, we have:

$$A_S\vec{E}_S + A_E\vec{E}_E = \vec{\nabla}_H\left[\Phi_1^e + V_2 + \delta\Phi_1^e + \delta d\,\partial_r(\Phi_1^e + V_2)\right] \tag{60}$$

that-is-to-say, the horizontal attraction is equal to the horizontal gradient of the perturbed geoid.

4.4. PERTURBATION OF SOME GEODETIC DEFORMATIONS

4.4.1. *Tilt*

Noting $\vec{\Delta}$ the incremental spherical angle between the instantaneous outward geometrical normal $\vec{n}_t$ and the instantaneous outward gravity normal $\vec{n}_g$ in the ellipsoidal basis defined in (47), Wahr (1981) defines the tilt in the east direction (or φ) by: $\Delta_E = \vec{e}_\varphi\cdot\vec{\Delta}$ and that in the other horizontal direction (approximatively south) by $\Delta_S = \vec{E}_S\cdot\vec{\Delta}$. Note that for an elliptical Earth, the horizontal vector $\vec{E}_S$ is not quite equal to $\vec{e}_\theta$ (see Equation (47)). For the spheroidal incompressible homogeneous Earth model, we obtain:

$$\begin{aligned}\Delta_S &= \frac{V_o}{g_o a}\left[(\gamma_2+\delta\gamma_2)\partial_\theta P_2^2 + \delta\gamma_4\partial_\theta P_4^2\right]\cos(\sigma t + 2\varphi),\\ \Delta_E &= \frac{V_o}{g_o a}\left[(\gamma_2+\delta\gamma_2)P_2^2 + \delta\gamma_4 P_4^2\right]\frac{1}{\sin\theta}\partial_\varphi\cos(\sigma t + 2\varphi)\end{aligned} \tag{61}$$

with

$$\begin{aligned}\gamma_2 &= 1 + k_2 - h_2,\\ \delta\gamma_2 &= \delta k_2 - \delta h_2 + \frac{4}{21}\alpha\left[1 + k_2 + 3h_2 - 6l_2\right],\\ \delta\gamma_4 &= \delta k_4 - \delta h_4 - \frac{2}{35}\alpha\left[1 + k_2 + 3h_2 - 6l_2\right].\end{aligned} \tag{62}$$

For our incompressible homogeneous Earth model, we have:

$$\gamma_2 = 1 - \frac{1}{1+\bar{\mu}}(1+\delta X_2),\quad \delta\gamma_2 = -\frac{\alpha}{1995}\frac{1805 + 33\bar{\mu} - 380\bar{\mu}^2}{(1+\bar{\mu})^2},$$
$$\delta\gamma_4 = -\frac{\alpha}{105}\frac{114 + 857\bar{\mu} + 306\bar{\mu}^2}{(1+\bar{\mu})(51\bar{\mu}+38)}$$

with the numerical values given in Section 3.2, we obtain:

$$\gamma_2 + \delta\gamma_2 = 0.759614,\quad \delta\gamma_4 = -0.0002913.$$

For the semi-diurnal tidal wave M_2, the order of magnitude of the perturbations will be: $(\gamma_2+\delta\gamma_2)\frac{V_o}{g_o a}=2.18$ mas and $\delta\gamma_4\frac{V_o}{g_o a}=-0.8\mu$ as.

If the planet is rigid, these tilt coefficients will become:

$$\gamma_2=1, \quad \delta\gamma_2=\frac{4}{21}\alpha=0.0008210, \quad \delta\gamma_4=-\frac{2}{35}\alpha=-0.0002463.$$

4.4.2. *Changes of the Vertical with Reference to the Earth's Axis or to a Fixed Direction*

For astronomical instruments, the deflection of the vertical at a point cause changes in the astronomic coordinates. For instruments related to the outward normal to the geoid (for example with a bath of mercury), the latitude is determined by comparing the direction of the vertical to the direction of the axis of rotation of the Earth. We can express from Equations (45) and (49) the respective components of the deflection of the vertical $\vec{n}_g$ and of the tangential displacement $\vec{u}_H$ in the ellipsoidal basis (47). The perturbation of the angle related to the deflection of the vertical with respect to the Earth's rotational axis will be:

$$\left(\vec{n}_g,\vec{e}_z\right)-\left(\vec{E}_n,\vec{e}_z\right)=\vec{n}_g-\vec{E}_n+\vec{u}_H. \tag{63}$$

The perturbation of this direction may be written using the previous Love numbers. We have:

$$\begin{aligned}\left(\vec{n}_g,\vec{E}_n\right)=&-\frac{V_o}{g_o a}\Big\{\left[(\Lambda_2+\delta\Lambda_2)\partial_\theta P_2^2+\delta\Lambda_4\partial_\theta P_4^2\right]\cos(\sigma t+2\varphi)\\&\qquad-\delta\Lambda_3\frac{1}{\sin\theta}P_3^2\partial_\varphi\sin(\sigma t+2\varphi)\Big\}\vec{E}_S\\&-\frac{V_o}{g_o a}\Big\{\left[(\Lambda_2+\delta\Lambda_2)P_2^2+\delta\Lambda_4 P_4^2\right]\frac{1}{\sin\theta}\partial_\varphi\cos(\sigma t+2\varphi)\\&\qquad+\delta\Lambda_3\partial_\theta P_3^2\sin(\sigma t+2\varphi)\Big\}\vec{E}_E\end{aligned} \tag{64}$$

with:

$$\begin{aligned}&\Lambda_2=1+k_2-l_2,\\&\delta\Lambda_2=\delta k_2-\delta l_2+\frac{2}{21}\alpha[2+2k_2-l_2],\\&\delta\Lambda_4=\delta k_4-\delta l_4-\frac{1}{35}\alpha[2+2k_2-l_2],\\&\delta\Lambda_3=-\Delta l_*.\end{aligned} \tag{65}$$

For our incompressible homogeneous Earth's model:

$$\Lambda_2 = 1 + \frac{3}{4}\frac{1}{1+\bar{\mu}}(1+\delta X_2), \quad \delta\Lambda_2 = \frac{\alpha}{1995}\frac{931+267\bar{\mu}+380\bar{\mu}^2}{(1+\bar{\mu})^2},$$
$$\delta\Lambda_4 = -\frac{\alpha}{1260}\frac{3458+5235\bar{\mu}+3672\bar{\mu}^2}{(1+\bar{\mu})(51\bar{\mu}+38)}$$

with the numerical values given in Section 3.2, we obtain:

$$\Lambda_2 + \delta\Lambda_2 = 1.181161, \quad \delta\Lambda_4 = -0.0002340.$$

For the semi-diurnal tidal wave M_2, the order of magnitude of the perturbations will be:

$$(\Lambda_2+\delta\Lambda_2)\frac{V_o}{g_o a} = 3.39\,\text{mas}, \quad \delta\Lambda_3\frac{V_o}{g_o a} = 0.6\,\mu\text{as}, \quad \text{and} \quad \delta\Lambda_4\frac{V_o}{g_o a} = -0.7\mu\text{as}.$$

For instruments related to the topographic outward normal (46) with respect to a fixed direction (for example V.L.B.I.) noted $\vec{d}$, we compute the changes in the angle $\left(\vec{n}_t, \vec{d}\right)$:

$$\left(\vec{n}_t, \vec{d}\right) - \left(\vec{E}_n, \vec{d}\right) = \vec{n}_t - \vec{E}_n + \vec{u}_H. \tag{66}$$

We may similarly define β coefficients such as:

$$\begin{aligned}\left(\vec{n}_t, \vec{E}_n\right) = &-\frac{V_o}{g_o a}\Big\{\left[(\beta_2+\delta\beta_2)\partial_\theta P_2^2 + \delta\beta_4\partial_\theta P_4^2\right]\cos(\sigma t+2\varphi) \\ &\qquad -\delta\beta_3\frac{1}{\sin\theta}P_3^2\partial_\varphi\sin(\sigma t+2\varphi)\Big\}\vec{E}_S \\ &-\frac{V_o}{g_o a}\Big\{\left[(\beta_2+\delta\beta_2)P_2^2 + \delta\beta_4 P_4^2\right]\frac{1}{\sin\theta}\partial_\varphi\cos(\sigma t+2\varphi) \\ &\qquad +\delta\beta_3\partial_\theta P_3^2\sin(\sigma t+2\varphi)\Big\}\vec{E}_E\end{aligned} \tag{67}$$

with

$$\begin{aligned}&\beta_2 = h_2 - l_2, \\ &\delta\beta_2 = \delta h_2 - \delta l_2 - \frac{4}{7}\alpha h_2 + \frac{22}{21}\alpha l_2, \\ &\delta\beta_4 = \delta k_4 - \delta l_4 + \frac{1}{35}\alpha[6h_2 - 11l_2], \\ &\delta\beta_3 = -\Delta l_*.\end{aligned} \tag{68}$$

For our incompressible homogeneous Earth's model:

$$\beta_2 = \frac{7}{4}\frac{1}{1+\bar{\mu}}(1+\delta X_2), \quad \delta\beta_2 = \frac{4\alpha}{665}\frac{228+25\bar{\mu}}{(1+\bar{\mu})^2},$$
$$\delta\beta_4 = \frac{\alpha}{1260}\frac{11590+17697\bar{\mu}}{(1+\bar{\mu})(51\bar{\mu}+38)}$$

with the numerical values given in Section 3.2, we obtain:

$$\beta_2 + \delta\beta_2 = 0.421547, \quad \delta\beta_4 = 0.0002786.$$

For the semi-diurnal tidal wave M_2, the order of magnitude of the perturbations will be:

$$(\beta_2 + \delta\beta_2)\frac{V_o}{g_o a} = 1.21\,\text{mas}, \quad \delta\beta_3\frac{V_o}{g_o a} = 0.6\mu\text{as}, \quad \text{and } \delta\beta_4\frac{V_o}{g_o a} = 0.8\mu\text{as}.$$

5. Conclusion

We have presented the analytical elasto-gravitational solutions for an incompressible homogeneous spheroidal hydrostatic pre-stress planet submitted to the semi-diurnal tidal potential.

We have pointed out the problem related to the spherical reference model which takes or not into account the radial fluid deformation induced by the radial part of the centrifugal potential. As PREM is a mean spherical model built from seismological observations, we think that these effects are already taken into account in the geometrical and physical parameters of the initial reference sphere. Therefore, we believe that our results for $\delta h_2, \delta l_2$, and δk_2 are more realistic.

We have extended these results to the analytical study of the geodetic and gravimetric response of the Earth to the tides on the spheroidal deformed surface. The order of magnitude of the perturbations of the displacement is the millimeter and is consequently significant, because it is well known that for space geodesy, it is now necessary to achieve the mm level in the tidal displacements. The order of magnitude of the perturbation of the direction of the vertical is the micro arc-sec that is to say too small to be detected using VLBI. As a matter of fact, VLBI determinations of earth-rotation variations, and of the coordinates of terrestrial sites and celestial objects are made currently with estimated accuracies of about ± 0.2 milliarcsecond or better.

That simple analytical solutions are not realistic but they are useful to understand the physics of the problem, and especially the influence of each geometric and physical parameter. They have also been used to check the numerical procedure of a new method (a spectral element method) developed in order to take into account lateral variations of density and rheological parameters, deviatoric pre-stresses and interfaces topography (Métivier, 2005). We have developed such a model, because with the new generation of gravity measurements, one of the challenges of the future 10-years will be to provide more realistic Earth time gravity variation models. Realistic solid tide models notably are needed for global consideration with gravity satellites like GRACE, GOCE, and in the future GRACE/GOCE follow on. More realistic gravity variation models are also needed for local and surface measurements, particularly with the emergence of gravimetric observatories network like the GGP network (Global Geodynamic Project) which uses very accurate superconducting gravimeters.

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A. Boundary Conditions for the Perturbed System and Solutions

The constants $\delta C_3, \delta C_4, \delta C_5$, and δC_7 introduced in the perturbed δy_i propagator (see 36) may be determined from the boundary conditions (27).

We use the spherical harmonics Y_n^{mc} and Y_n^{ms}. Consequently, we have, for each degree n and order m, to determine the eight constants $(\delta C_3)_n^{mc}$, $(\delta C_3)_n^{ms}$, $(\delta C_4)_n^{mc}$, $(\delta C_4)_n^{ms}$, $(\delta C_5)_n^{mc}$, $(\delta C_5)_n^{ms}$, $(\delta C_7)_n^{mc}$, and $(\delta C_7)_n^{ms}$ from the boundary conditions.

A.1. RADIAL ATTRACTION

At the Earth's surface, we have, for the radial attraction:

$$[\delta\xi_r]_{a^-}^{a^+} = \left[-\delta d\,\partial_r \xi_r + \vec{\nabla}^S \delta d \cdot \vec{\xi}\right]_{a^-}^{a^+}. \tag{A1}$$

From the y_i system (16) and the associated boundary conditions for a spherical Earth, it is easy to show that:

$$\left[-\delta d\ \partial_r \xi_r + \vec{\nabla}^S \delta d \cdot \vec{\xi}\right]_{a^-}^{a^+} = -2\alpha g_o \frac{y_3(a)}{a}\left[n(n+1) Y_n^m P_2^0 + P_2^1 \partial_\theta Y_n^m\right]. \tag{A2}$$

For a degree 2-order 2 potential, we have the following relations:

$$P_2^0\ P_2^2 = -\tfrac{2}{7} P_2^2 + \tfrac{3}{35} P_4^2 \quad \text{and} \quad P_2^1 P_2^1 = \tfrac{3}{7} P_2^2 + \tfrac{6}{35} P_4^2 \tag{A3}$$

with the Legendre polynomials: $P_2^2 = 3\sin^2\theta$ and $P_4^2 = \frac{15}{2}\sin^2\theta(7\cos^2\theta - 1)$.

Consequently, the Equation (A2) for the semi-diurnal tidal M_2 luni-solar potential becomes:

$$\left[-\delta d\ \partial_r\xi_r+\vec{\nabla}^S\delta d\cdot\ \vec{\xi}\right]_{a^-}^{a^+}=-2\alpha l_2\frac{V_o\cos(\sigma t+2\varphi)}{a}\frac{6}{7}\left[-P_2^2+P_4^2\right]. \qquad \text{(A4)}$$

The left-hand side of the Boundary Condition (A1) may be written:

$$\left[-\frac{\partial\delta\Phi_1^e}{\partial r}\right]_{a^+}-\delta\xi_r(a) \qquad \text{(A5)}$$

with

$$\delta\xi_r=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left\{-n(\delta C_5)_n^m r^{n-1}+\tfrac{3g_o}{a}\left[(\delta C_3)_n^m r^{n+1}+(\delta C_4)_n^m r^{n-1}\right]\right\}Y_n^m \text{ and}$$
$$\left[-\frac{\partial\delta\Phi_1^e}{\partial r}\right]_{a^+}=\sum_{n,m}\frac{(n+1)}{a}\left(\delta\Phi_1^e\right)_n^m(a^+)Y_n^m(\theta,\varphi).$$

From the continuity of the potential, we have:

$$\left[\delta\Phi_1^e\right]_{a^-}^{a^+}=-\delta d\ \left[\partial_r\Phi_1^e\right]_{a^-}^{a^+}=+4\pi G\rho_o u_r\ \delta d \qquad \text{(A6)}$$

Consequently, we have:

$$\begin{aligned}\left(\delta\Phi_1^e\right)_n^m(a^+)&=\left(\delta\Phi_1^e\right)_n^m(a)+2\alpha h_2 V_o\cos(\sigma t+2\varphi)\\&\times\left[\frac{2}{7}P_2^2\ \delta_n^2\delta_m^2-\frac{3}{35}P_4^2\ \delta_n^4\delta_m^2\right],\end{aligned} \qquad \text{(A7)}$$

where δ_i^j is the Kronecker symbol ($\delta_i^j=1$ if $i=j$ and $\delta_i^j=0$ if $i\neq j$).

The spherical harmonics Y_n^{mc} and Y_n^{ms} define a basis and consequently, we can equal each coefficients of degree n and order m of the right- and left-hand side of the boundary condition (A1); it leads to four equations:

$$\begin{aligned}
&5\left(\delta C_5^c\right)_2^2 a-3g_o\left[\left(\delta C_3^c\right)_2^2a^2+\left(\delta C_4^c\right)_2^2\right]=-3\frac{\alpha V_o}{a(1+\bar{\mu})}\cos(\sigma t),\\
&5\left(\delta C_5^s\right)_2^2 a-3g_o\left[\left(\delta C_3^s\right)_2^2a^2+\left(\delta C_4^s\right)_2^2\right]=3\frac{\alpha V_o}{a(1+\bar{\mu})}\sin(\sigma t),\\
&9\left(\delta C_5^c\right)_4^2 a^3-3g_oa^2\left[\left(\delta C_3^c\right)_4^2a^2+\left(\delta C_4^c\right)_4^2\right]=\frac{6}{7}\frac{\alpha V_o}{a(1+\bar{\mu})}\cos(\sigma t),\\
&9\left(\delta C_5^s\right)_4^2 a^3-3g_oa^2\left[\left(\delta C_3^s\right)_4^2a^2+\left(\delta C_4^s\right)_4^2\right]=-\frac{6}{7}\frac{\alpha V_o}{a(1+\bar{\mu})}\sin(\sigma t).
\end{aligned} \qquad \text{(A8)}$$

A.2. TRACTIONS

The boundary condition for the surface tractions may be written from the perturbed Cauchy stress tensor:

$$\left[\vec{e}_r \cdot \overline{\overline{\delta\Sigma_1^l}}\right]_{r=a} = \left[-\delta d\ \vec{e}_r \cdot \partial_r \overline{\overline{\Sigma_1^l}} + \vec{\nabla}^S \partial d \cdot \overline{\overline{\Sigma_1^l}}\right]_{a^-}^{a^+}. \tag{A9}$$

The left-hand side of this equation may be written from (32)

$$\begin{aligned}\left[\vec{e}_r\ .\ \overline{\overline{\delta\Sigma_1^l}}\right]_{r=a} &= \vec{\delta}T^1(a)\\ &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \delta y_{2n}^m(a) Y_n^m(\theta,\varphi)\vec{e}_r + a\delta y_{4n}^m(a)\vec{\nabla}Y_n^m(\theta,\varphi)\\ &\quad + a\delta y_{8n}^m(a)\vec{\nabla}\Lambda\vec{e}_r Y_n^m(\theta,\varphi).\end{aligned} \tag{A10}$$

The right-hand side of (32) may be expanded into spheroidal and toroidal vector:

$$\begin{aligned}&\begin{pmatrix} -\delta d\ \left[\partial_r(\Sigma_{rr}^l)\right]_{r=a} + \frac{1}{a}\partial_\theta(\delta d)\Sigma_{r\theta}^l(a)\\ -\delta d\ \left[\partial_r(\Sigma_{r\theta}^l)\right]_{r=a} + \frac{1}{a}\partial_\theta(\delta d)\Sigma_{\theta\theta}^l(a)\\ -\delta d\ \left[\partial_r(\Sigma_{r\varphi}^l)\right]_{r=a} + \frac{1}{a}\partial_\theta(\delta d)\Sigma_{\theta\varphi}^l(a)\end{pmatrix} = \frac{1}{19}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}\\ &\times\left\{-\frac{2}{3}(32\bar{\mu}+57)\left(\frac{-2}{7}P_2^2 + \frac{3}{35}P_4^2\right)\cos(\sigma t+2\varphi)\vec{e}_r\right.\\ &\quad + 2\bar{\mu}\vec{\nabla}^S\left[\left(\frac{37}{21}P_2^2 - \frac{8}{35}P_4^2\right)\cos(\sigma t+2\varphi)\right]\\ &\quad \left. + \frac{4}{15}\bar{\mu}\vec{e}_r^{\wedge}\vec{\nabla}^S\left[P_3^2\sin(\sigma t+2\varphi)\right]\right\}.\end{aligned} \tag{A11}$$

The boundary conditions in tractions may be consequently written: For $\delta y_4(a)$:

$$\begin{aligned}\left[\frac{4}{3}\left(\delta C_3^c\right)_2^2 a^2 + \frac{1}{2}\left(\delta C_4^c\right)_2^2\right] &= \frac{37}{42}\frac{\alpha V_o}{g_o a(1+\bar{\mu})}\cos\sigma t,\\ \left[\frac{4}{3}\left(\delta C_3^s\right)_2^2 a^2 + \frac{1}{2}\left(\delta C_4^s\right)_2^2\right] &= -\frac{37}{42}\frac{\alpha V_o}{g_o a(1+\bar{\mu})}\sin\sigma t,\\ a^2\left[\frac{6}{5}\left(\delta C_3^c\right)_4^2 a^2 + \frac{3}{4}\left(\delta C_4^c\right)_4^2\right] &= -\frac{4}{35}\frac{\alpha V_o}{g_o a(1+\bar{\mu})}\cos\sigma t,\\ a^2\left[\frac{6}{5}\left(\delta C_3^s\right)_4^2 a^2 + \frac{3}{4}\left(\delta C_4^s\right)_4^2\right] &= +\frac{4}{35}\frac{\alpha V_o}{g_o a(1+\bar{\mu})}\sin\sigma t.\end{aligned} \tag{A12}$$

For $\delta y_8(a)$:

$$a^2\left(\delta C_7^c\right)_3^2 = -\frac{1}{15}\frac{\alpha V_o}{g_o a(1+\bar{\mu})}\sin\ \sigma t,$$
$$a^2\left(\delta C_7^s\right)_3^2 = -\frac{1}{15}\frac{\alpha V_o}{g_o a(1+\bar{\mu})}\cos\ \sigma t. \tag{A13}$$

For $\delta y_2(a)$, taking into account the x_2^2 and x_4^2 coefficients defined in (30):

$$-\rho_o a^2\left(\delta C_5^c\right)_2^2 + \frac{\rho_o g_o a}{19}\left[(19-2\bar{\mu})a^2\left(\delta C_3^c\right)_2^2 + (19+4\bar{\mu})\left(\delta C_4^c\right)_2^2\right]$$
$$= \frac{1}{399}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}(128\bar{\mu}+779)\cos(\sigma t),$$
$$-\rho_o a^2\left(\delta C_5^s\right)_2^2 + \frac{\rho_o g_o a}{19}\left[(19-2\bar{\mu})a^2\left(\delta C_3^s\right)_2^2 + (19+4\bar{\mu})\left(\delta C_4^s\right)_2^2\right]$$
$$= -\frac{1}{399}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}(128\bar{\mu}+779)\sin(\sigma t), \tag{A14}$$
$$-\rho_o a^4\left(\delta C_5^c\right)_4^2 + \frac{\rho_o g_o a^3}{19}\left[(19+9\bar{\mu})a^2\left(\delta C_3^c\right)_4^2 + (19+12\bar{\mu})\left(\delta C_4^c\right)_4^2\right]$$
$$= -\frac{2}{665}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}(32\bar{\mu}+95)\cos(\sigma t),$$
$$-\rho_o a^4\left(\delta C_5^s\right)_4^2 + \frac{\rho_o g_o a^3}{19}\left[(19+9\bar{\mu})a^2\left(\delta C_3^s\right)_4^2 + (19+12\bar{\mu})\left(\delta C_4^s\right)_4^2\right]$$
$$= \frac{2}{665}\frac{\alpha\rho_o V_o}{1+\bar{\mu}}(32\bar{\mu}+95)\sin(\sigma t).$$

A.3. SOLUTIONS

We solve the 14 equations in order to find the 14 unknowns:

$$\left(\delta C_3^c\right)_2^2 = \frac{2}{665}\frac{\alpha V_o}{a^3 g_o}\frac{(25\bar{\mu}-323)\cos(\sigma t)}{(1+\bar{\mu})^2}, \quad \left(\delta C_3^s\right)_2^2 = -\frac{2}{665}\frac{\alpha V_o}{a^3 g_o}\frac{(25\bar{\mu}-323)\sin(\sigma t)}{(1+\bar{\mu})^2},$$
$$\left(\delta C_4^c\right)_2^2 = \frac{1}{1995}\frac{\alpha V_o}{a g_o}\frac{(3115\bar{\mu}+8683)\cos(\sigma t)}{(1+\bar{\mu})^2}, \quad \left(\delta C_4^s\right)_2^2 = -\frac{1}{1995}\frac{\alpha V_o}{a g_o}\frac{(3115\bar{\mu}+8683)\sin(\sigma t)}{(1+\bar{\mu})^2},$$
$$\left(\delta C_5^c\right)_2^2 = \frac{2}{665}\frac{\alpha V_o}{a^2}\frac{(127\bar{\mu}+475)\cos(\sigma t)}{(1+\bar{\mu})^2}, \quad \left(\delta C_5^s\right)_2^2 = -\frac{2}{665}\frac{\alpha V_o}{a^2}\frac{(127\bar{\mu}+475)\sin(\sigma t)}{(1+\bar{\mu})^2},$$
$$\left(\delta C_3^c\right)_4^2 = \frac{76}{9}\frac{\alpha V_o}{a^5 g_o}\frac{\cos(\sigma t)}{(1+\bar{\mu})(51\bar{\mu}+38)}, \quad \left(\delta C_3^s\right)_4^2 = -\frac{76}{9}\frac{\alpha V_o}{a^5 g_o}\frac{\sin(\sigma t)}{(1+\bar{\mu})(51\bar{\mu}+38)},$$
$$\left(\delta C_4^c\right)_4^2 = -\frac{16}{315}\frac{\alpha V_o}{a^3 g_o}\frac{(153\bar{\mu}+380)\cos(\sigma t)}{(1+\bar{\mu})(51\bar{\mu}+38)}, \quad \left(\delta C_4^s\right)_4^2 = \frac{16}{315}\frac{\alpha V_o}{a^3 g_o}\frac{(153\bar{\mu}+380)\sin(\sigma t)}{(1+\bar{\mu})(51\bar{\mu}+38)},$$
$$\left(\delta C_5^c\right)_4^2 = \frac{34}{15}\frac{\alpha V_o}{a^4}\frac{\bar{\mu}\cos(\sigma t)}{(1+\bar{\mu})(51\bar{\mu}+38)}, \quad \left(\delta C_5^s\right)_4^2 = -\frac{34}{15}\frac{\alpha V_o}{a^4}\frac{\bar{\mu}\sin(\sigma t)}{(1+\bar{\mu})(51\bar{\mu}+38)},$$
$$\left(\delta C_7^c\right)_3^2 = -\frac{1}{15}\frac{\alpha V_o}{g_o a^3}\frac{\sin(\sigma t)}{(1+\bar{\mu})}, \quad \left(\delta C_7^s\right)_3^2 = -\frac{1}{15}\frac{\alpha V_o}{g_o a^3}\frac{\cos(\sigma t)}{(1+\bar{\mu})}.$$

Knowing the constants, we know the perturbations of the displacement, of the gravitational potential and of the stress tensor within the entire planet.

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