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Practical method for drawing a VGP path

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ABSTRACT

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A statistical method is proposed for the smoothing of polar wander paths and for giving their confidence limits by the mean of successive ellipses. The method is fully parametrical and is based on the relations between the inertia matrix and the parameters of the Fisher distribution, from which a bivariate form is deduced. An elementary tensorial calculation gives the parameters of the confidence ellipse around a vectorial weighted mean, for any unimodal set of vectors. This model can also be used for other statistical tests, wherever the rotational symmetry hypothesis is not consistent (i.e. the fold test).

1. Introduction

Drawing a polar wander path requires the determination of the averaged direction of a weighted sum of vectorial means derived from Fisher's spherical distribution. Several methods have already been proposed (Van Alstine and De Boer, 1979; Thompson and Clark, 1981; Harrison and Lindh, 1982; Irving and Irving, 1982). The new one, developed by Le Goff (1990), is elaborated around a bivariate extension of the Fisher distribution. The parameters of this model are estimated directly from the terms of the inertia tensor which can be associated to any vectorial set. All of these vectorial sets can be of miscellaneous kinds: unit vectors, weighted vectors, weighted Fisher means or inertia tensors. We may thus easily introduce many types of weighting for any chronological spherical path, and propose this 'Fisher-like' model as a test for the rotational symmetry hypothesis.

After analysis of the mathematical model, some examples of the practical use of the method are given.

2. Mathematical model

The mathematical development is done in geographical spherical coordinates (colatitude θ and longitude φ) in an orthonormal system (Ox, Oy, Oz).

(1) As for paleomagnetic data, the basic assumption is a strong concentration around the mean direction (θ_0, φ_0) of N vectors (θ_i, φ_i) bearing a unit mass m_i at their end.

First, let us consider that the dispersion follows the Fisherian probability model around the Oz axis ($\theta_0 = 0$): $F(0, \kappa) = [\kappa/4\pi \ \mathrm{sh}(\kappa)] \exp(\kappa \cos \theta)$ (Fisher, 1953). With $z = \cos \theta$,

$x = \sin\theta\cos\varphi$ and $y = \sin\theta\sin\varphi$, the inertia tensor, which here is diagonal, can be written:

$$\boldsymbol{D} = \begin{pmatrix} N-\sum x_i^2 & -\sum x_i y_i & -\sum x_i z_i \\ -\sum x_i y_i & N-\sum y_i^2 & -\sum y_i z_i \\ -\sum x_i z_i & -\sum y_i z_i & N-\sum z_i^2 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \quad (1)$$

where $A+B+C=2N$ and $A=B$.

Using the approximations $\theta \approx \sin\theta$ and $1-\theta^2/2 \approx \cos\theta$, the relations between the diagonal terms (A, B, C) of $\boldsymbol{D}$ (1) and the estimator $k = N/(N-\Sigma\cos\theta_i)$ of the Fisher concentration parameter κ (Fisher, 1953; Mardia, 1972; Watson, 1983) can be established:

$$C = \frac{2N}{1+k}$$

and

$$A = B = \frac{kN}{1+k} \quad (2)$$

By many samplings, the validity limits of the relations of eqn. (2) have been proved at 3% departure for $k > 8$ and at 0.5% for $k > 30$ whenever $N > 10$.

In a second step, all the terms of the tensor $\boldsymbol{I}$, in the general case of a mean direction (paleopole) of latitude λ and longitude φ, are determined by application of the following rotation $\boldsymbol{R}$.

$$\boldsymbol{R} = \begin{pmatrix} \sin\lambda\cos\varphi & \sin\lambda\sin\varphi & -\cos\varphi \\ -\sin\varphi & \cos\varphi & 0 \\ \cos\lambda\cos\varphi & \cos\lambda\sin\varphi & \sin\lambda \end{pmatrix}$$

and

$$\boldsymbol{I} = \boldsymbol{R}'\boldsymbol{D}\boldsymbol{R} \quad (3)$$

(2) The problem of the weighted summation of several populations, each characterized by a tensor $\boldsymbol{I}_j$, can thus be solved as a tensorial addition ($\boldsymbol{I}_t = \Sigma w_j \boldsymbol{I}_j$), w_j being the weight given to the jth population. The minimal eigen direction $(\lambda_{min}, \varphi_{min})$ of $\boldsymbol{I}_t$ is then a good estimator of the mean direction of this new population (Watson, 1983), but the tensor has now lost its rotational symmetry ($A_t \neq B_t$).

(3) Let the diagonal tensor of $\boldsymbol{I}_t$ be $\boldsymbol{D}_t$, the eigen directions of which being Ox, Oy, Oz, and keep the hypothesis of small scattering around Oz ($\theta_0 = 0$). Remember that, now, $N_t = \Sigma w_j$ and, for reading clarity, the subscript t will be removed ($N = N_t$, $A = A_t, \ldots$). Let us apply the Gauss distribution in the (Ox, Oy) projection plane of the end of the (fictive) vectors (θ_i, φ_i, $m_i = 1$) representing the new non-rotational population:

$$G(0, \sigma_x, \sigma_y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp-\left(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right) \quad (4)$$

It appears that the variances of eqn. (4) $\sigma_x^2 = \Sigma x_i^2/N$ and $\sigma_y^2 = \Sigma y_i^2/N$ are already expressed in the tensor $\boldsymbol{D}$ (1): $A = N - \Sigma x_i^2 = N(1-\sigma_x^2)$. Since $A + B \approx 2N$, this leads to the relations

$$k_x = \frac{1}{\sigma_x^2} = \frac{(B+A)}{(C+B-A)}$$

$$k_y = \frac{1}{\sigma_y^2} = \frac{(B+A)}{(C+A-B)} \quad (5)$$

which give, as a function of the eigenvalues (A, B, C) of the inertia tensor, the parameters k_x and k_y, reciprocal of the variances in the plane.

Multiplying the numerator and denominator of eqn. (4) by $\exp(k_x\cos^2\varphi + k_y\sin^2\varphi)$ and using $x = \theta\cos\varphi$, $y = \theta\sin\varphi$, and $(1-\theta^2/2) \approx \cos\theta$, we may write:

$$P(0, k_x, k_y) = \frac{\sqrt{(k_x k_y)}}{2\pi\exp\left(k_x\cos^2\varphi + k_y\sin^2\varphi\right)} \times\exp\left(k_x\cos^2\varphi + k_y\sin^2\varphi\right)\cos\theta \quad (6)$$

which is a particular form of Fisher statistic where k is a function of φ.

(4) These developments lead us to propose a bivariate probability function which can be seen

as the unimodal form of the Bingham distribution (Onstott, 1980):

$$P(0, \kappa_x, \kappa_y) = C^{-1}_{(\kappa_x, \kappa_y)} \exp\left[\left(\kappa_x \cos^2 \varphi + \kappa_y \sin^2 \varphi\right) \cos \theta\right] \quad (7)$$

The successive eqns. (1)–(7) clearly show that κ_x and κ_y may be estimated, in the usual case of small scattering, by k_x and k_y of the formulae of eqn. (5). In the general case of any dispersion on the sphere, the theory of this probability function has yet to be elaborated to define the best estimators of its parameters, and the exact confidence surface around the mean direction. For paleomagnetic usage, Le Goff (1990), after integrating $C(\kappa_x, \kappa_y)$, showed and verified by numerical calculations, using formulae similar to that of the Fisher case, that the following relations are well adapted for the computation of the 95% confidence ellipse axes around the mean direction:

$$\alpha_{95}(x) = \frac{140}{\sqrt{\left(k_x \sum w_j N_j\right)}} (^\circ)$$

and

$$\alpha_{95}(y) = \frac{140}{\sqrt{\left(k_y \sum w_j N_j\right)}} (^\circ) \quad (8)$$

The orientation of the ellipse, with respect to the meridian crossing the mean direction, is given, from the eigen directions of $\boldsymbol{I}_t$, by the angle Ω (Fig. 1(a)):

$$\Omega = \frac{\varphi_{int} - \varphi_{min}}{|\varphi_{int} - \varphi_{min}|} \text{Arc} \cos \frac{\sin \lambda_{int}}{\cos \lambda_{min}},$$

$$\text{with } \lambda_{int} \text{ like that } |\varphi_{int} - \varphi_{min}| < \pi \quad (9)$$

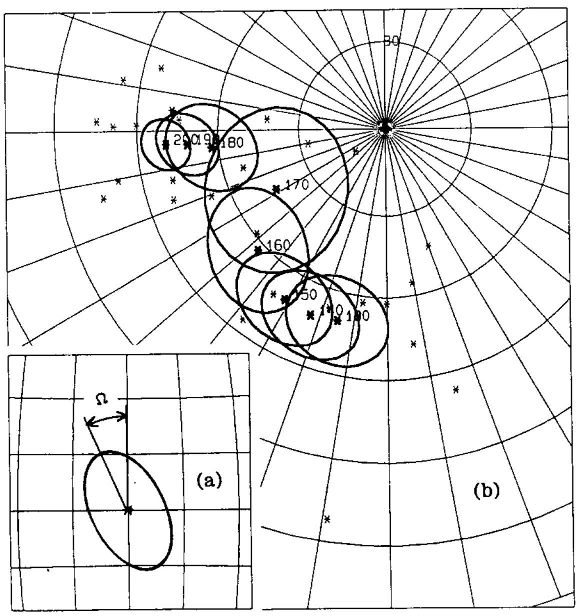

Fig. 1. (a) Definition of Ω (eqn. (9)). (b) Successive 95% confidence ellipses showing the polar wander path between 130 and 200 Ma for the North American continent, obtained with the Harrison and Lindh (1982) selection. The 27 VGP (*), intercepted by a 30 Ma sliding window, are weighted by their age incertitude interval combined with the authors' weighting factor.

3. Recapitulation of the practical method for the calculation of the parameters of a weighted sum of Fisherian distributions

Let us consider a set of Fisherian distributions, like VGP, known by five parameters: the latitude λ_j, the longitude φ_j, the concentration parameter k_j, the number of samples or sites N_j ($N_j = 1$ if this number is not an element of weighting) and the statistical weight w_j for this jth distribution (for the apparent polar wander path (APWP), w_j can be in a function of the age uncertainty):

(1) A_j, B_j, C_j of the tensor $\boldsymbol{D}_j$ are calculated using eqn. (2).

(2) The relations of eqn. (3) allow $\boldsymbol{R}_j$, $\boldsymbol{R}_j'$ and $\boldsymbol{I}_j = \boldsymbol{R}_j' \boldsymbol{D}_j \boldsymbol{R}_j$ to be obtained; then the weighted tensor is $\boldsymbol{I}_t = \sum w_j \boldsymbol{I}_j$.

(3) The mean direction (λ_{min}, φ_{min}) and the other eigen directions are calculated by diagonalization of $\boldsymbol{I}_t$. The terms A_t, B_t and C_t are known.

(4) k_x and k_y are obtained by eqn. (5), then $\alpha_{95}(x)$ and $\alpha_{95}(y)$ by eqn. (8) and finally the angle Ω by eqn. (9).

In the case of a set of unit vectors (i.e. magnetization directions), a tensor $\boldsymbol{I}$ (not diagonal) can be calculated by eqn. (1) and only steps (3) and (4) are to be followed.

4. Applications

In addition to the application for drawing a paleomagnetic apparent polar wander path such as in Fig. 1(b), drawn using the Harrison and Lindh data (1982), this bivariate unimodal distribution is a new easy tool for other directional tests. Care should be taken in the use of this method, by keeping in mind the initial assumption of small scattering, and above all that the use of tensorial calculations can cause axis reversals. Nevertheless, in the case where both reversed and normal directions are present, the use of this tensorial method is the best way to supply together the mean direction (Onstott, 1980; Fisher et al., 1987) and the 'Fisher-like' precision parameters without the need for tables or factors of conversion.

As in the comparison of two dispersions, the table of variance ratios (k_x/k_y) for a set of N samples, $F_{2(N-1),2(N-1)}$, can be used, to test the significance level of the rotational symmetry hypothesis.

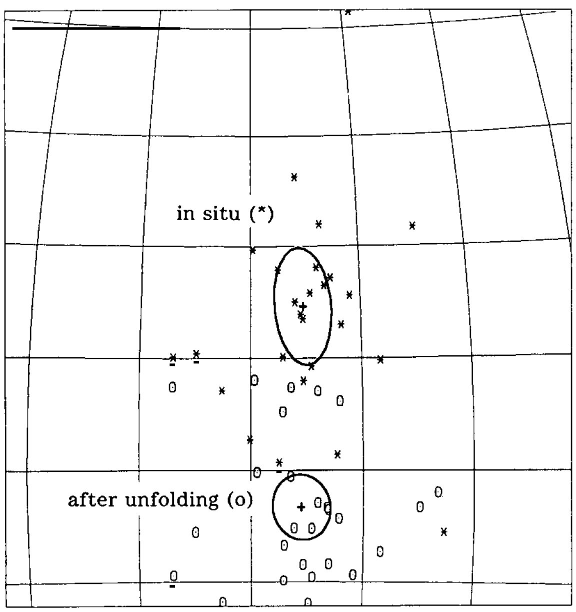

Fig. 2. Example of unfolding: the in situ confidence ellipse is elongated perpendicularly to the fold mean axis whereas, after unfolding, the rotational asymmetry is not significant.

Figure 2 shows the unfolding of 27 remanent magnetic directions of a Permian formation of Saint-Affrique (Diego Orozco, 1990). The distribution of the in situ directions was elongated perpendicularly to the fold mean axis with $k_x/k_y = 4.1$ ($F_{52,52} \approx 1.6$), which is significantly non-rotationally symmetric. After dip correction, $k_x/k_y = 1.3$, which means that the rotational symmetry can not be rejected. The use of this method, in conjunction with a correlative fold test (Enkin, 1990; McFadden, 1990), provides valuable information on the completion or not of a dip correction.

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