COMMUN. MATH. SCI.
Vol. 21, No. 1, pp. 151–172

LOW-MACH TYPE APPROXIMATION OF THE NAVIER-STOKES SYSTEM WITH TEMPERATURE AND SALINITY FOR FREE SURFACE FLOWS*

LÉA BOITTIN†, MARIE-ODILE BRISTEAU‡, FRANÇOIS BOUCHUT§,
ANNE MANGENEY¶, JACQUES SAINTE-MARIE‖, AND FABIEN SOUILLɆ†

Abstract. We are interested in free surface flows where density variations coming, for example, from temperature or salinity differences, play a significant role in the hydrodynamic regime. In water, acoustic waves travel much faster than gravity and internal waves, hence the study of models arising from compressible fluid mechanics often requires a decoupling between these waves. Starting from the compressible Navier-Stokes system, we derive the so-called Navier-Stokes-Fourier system in an "incompressible" regime using the low-Mach scaling, hence filtering the acoustic waves, neglecting the density dependency on the fluid pressure but keeping its variations in terms of temperature and salinity. A slightly modified low-Mach asymptotics is proposed to obtain a model with thermo-mechanical compatibility. The case when the density depends only on the temperature is studied first. Then the variations of the fluid density with respect to temperature and salinity are considered, and it seems to be the first time that salinity dependency is considered in this low Mach limit. The obtained models conserve the mass of the fluid but not the volume and satisfy the second principle of thermodynamics.

Keywords. Navier-Stokes equations; compressible and incompressible fluids; free surface flows; variable density flows; low-Mach approximation.

AMS subject classifications. 35Q30; 35Q35; 76D05; 76N06.

1. Introduction

In oceans and lakes, one of the predominant driving forces is the difference in density, caused by salinity and temperature variations: increasing the salinity and lowering the temperature of a fluid both increase its density [1]. Oceans and lakes are stratified: the water density varies along the vertical direction. In the present work, we aim at describing and simulating variable density flows with free surface.

The Navier-Stokes equations – or possibly simplified versions of these equations – are the cornerstone of the modeling of hydrodynamics in lakes and oceans. Compared to situations where the density is constant, when considering variable density flows with free surface, two characteristics appear:

- the variations of the fluid density can lead to acoustic waves,
- the fluid density adds nonlinearities in the Navier-Stokes equations.

---

*Received: January 09, 2022; Accepted (in revised form): April 09, 2022. Communicated by Siddhartha Mishra.

†Inria Paris, 2 rue Simone Iff, CS 42112, 75589 Paris Cedex 12 and Sorbonne Université, Univ. Paris Diderot, SPC, CNRS, Laboratoire Jacques-Louis Lions (LJLL), F-75005 Paris (lboittin.a3@gmail.com). https://team.inria.fr/ange/team-members/jacques-sainte-marie/

‡Inria Paris, 2 rue Simone Iff, CS 42112, 75589 Paris Cedex 12 and Sorbonne Université, Univ. Paris Diderot, SPC, CNRS, LJLL, F-75005 Paris (Marie-Odile.Bristeau@inria.fr).

§Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050), CNRS, Univ. Gustave Eiffel, UPEC, F-77454, Marne-la-Vallée, France (Francois.Bouchut@u-pem.fr). https://perso.math.u-pem.fr/bouchut.francois/

¶University Paris City, Institut de Physique du Globe de Paris, 1 rue Jussieu, Paris F-75005, France (Anne.Mangeney@ipgp.fr). http://www.ipgp.fr/m̃angeney/

‖Inria Paris, 2 rue Simone Iff, CS 42112, 75589 Paris Cedex 12 and Sorbonne Université, Univ. Paris Diderot, SPC, CNRS, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris (Jacques.Sainte-Marie@inria.fr).

**Électricité de France (EDF) R&D, Laboratoire National d'Hydraulique et Environnement, 6 quai Watier, F-78400 Chatou, France (Fabien.Souille@gmail.com).

These two aspects are detailed hereafter.

The density variations are usually small and a common assumption in the case of geophysical flows is the Boussinesq approximation [2]. It is widely used to simplify the Navier-Stokes equations with variable density and consists in ignoring density variations in momentum conservation equations except in the buoyancy force term. The underlying assumptions are (1) the density variations are very small with respect to the mean density and (2) the vertical scale of the motion is small compared to the vertical scale of variations in density, and the density and pressure variations due to motion are not larger than the total static variations of pressure and density [3,4]. The consequences of the Boussinesq approximation are listed in [5]. Notably, under the Boussinesq approximation, the volume of the water is conserved, while the mass is not conserved. With this approximation, the density can be defined as a function of any given tracer, the temperature or a pollutant for instance. The Boussinesq approximation is the basis of many ocean models such as ICON [6], NEMO [7] and POM [8].

Several authors have shown the benefits of taking into account non-Boussinesq effects in lake and ocean models, either for the propagation of internal waves [9] or for sea level variations induced by expansion/contraction processes [4,10,11]. In the case of ocean water, density is in general a function of pressure, temperature and salinity. Equations of state for water based on experiments are proposed in [12,13].

Consequently, approaches to take into account the non-Boussinesq effects have been developed. A possibility is to adopt pressure coordinates [14,15]. The non-Boussinesq equations written in pressure coordinates are isomorphic to the Boussinesq equations in z-coordinates, which allows to use the same algorithm for the non-Boussinesq model as for the Boussinesq model. So far, however, this approach is not widely used in ocean modeling, though it is available in the code MITgcm ([16]). A review of the different types of vertical coordinates used for ocean modeling is given in [17]. In [5], a non-hydrostatic non-Boussinesq model is presented. A non-hydrostatic pressure anomaly is related to a compressible (non-Boussinesq) density anomaly. Yet the authors of [5] are primarily concerned with the simulation of acoustic waves and thermal dilation is not investigated numerically. This model has recently been included in CROCO ([18]). In the present work, we focus on the inclusion of dilation effects in the model.

The approach chosen here to propose a non-Boussinesq model is different from the previous ones. Starting from the compressible Navier-Stokes equations, we propose a formulation of the Navier-Stokes-Fourier system in the asymptotics of an "incompressible" fluid. The term "Fourier" refers here to Fourier's law of thermal conduction. For most geophysical flows, water can be considered as incompressible in the sense that the variation of its density with respect to the fluid pressure is small [12,13] and in this case, the acoustic waves can be filtered out, this is the so-called low Mach number limit. This is advantageous from the computational point of view because a restrictive condition must be imposed on the time step when the acoustic waves are included [19]. Even if in some very particular situations [20], a coupling between gravity and acoustic waves can occur in the context of fluid density variations, we consider here situations (in terms of water depth and wave frequency) when the phenomena are decoupled.

The incompressible limit of the Navier-Stokes-Fourier system has been extensively studied mathematically, see [21,22] and the references therein. The incompressible limit is a low-Mach approximation of the Navier-Stokes-Fourier system. It consists in neglecting the pressure dependency in the fluid state law (but not the temperature and salinity dependencies). This limit is singular and requires not necessarily intuitive scalings, in particular for numerical approximations [23]. Since the studied phenomena strongly couple mechanics (fluid motion and rheology) with thermodynamics (temperature and salinity variations), we pay close attention that the obtained models do not violate the second principle of thermodynamics. The most simple way of performing the incompressible limit yields a model for which the energy balance exhibits discrepancies with respect to the energy balance of the original compressible system. To recover an energy balance that is close to that of the original system, which is important for the robustness of numerical methods, some corrections terms are incorporated in the incompressible model. These do not alter the accuracy of the approximation. Instead of conserving the volume of fluid and not the mass (which is what the Boussinesq approximation implies), the model derived here strictly conserves the mass and is enriched by the thermohaline expansion effects i.e. the volume is no longer conserved. In particular the velocity is not divergence free, but the divergence of the velocity field is equal to a right-hand side involving temperature and salinity gradients, as well as viscous dissipation. Our incompressible models include the dependency of density on salinity additionally to the dependency on temperature, and it seems to be the first time that this is considered in detail, including Onsager's principles. The proposed model is close to the one in [24], yet it is rigorously derived here. While solving the model in [24] required inverting a non-linear system, the numerical resolution of the present model is more simple, see [25]. Moreover, the model in the present work is 3D while it was only 2D in [24].

A multilayer approximation of the Navier-Stokes-Fourier system with a numerical scheme – endowed with strong stability properties – and several numerical test cases will be presented in a forthcoming paper [25].

The paper is organized as follows. The incompressible Navier-Stokes-Fourier and Euler-Fourier models are introduced in Section 2 and their low Mach limits are studied in Section 3 including the situation where the temperature and the salinity are considered. Finally in Section 4, the formulation of the considered models with the hydrostatic assumption is given.

2. The 3D Navier-Stokes-Fourier system

2.1. The compressible Navier-Stokes-Fourier system.

We consider the classical compressible Navier-Stokes system describing a free surface gravitational flow over a bottom topography $z_b(x,y)$,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{U}) = 0, \tag{2.1}$$

$$\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla\cdot(\rho \mathbf{U}\otimes\mathbf{U}) + \nabla p - \nabla\cdot\sigma = \rho\mathbf{g}, \tag{2.2}$$

$$\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{U}|^2}{2} + \rho e\right) + \nabla\cdot\left(\left(\rho\frac{|\mathbf{U}|^2}{2} + \rho e + p - \sigma\right)\mathbf{U}\right) = -\nabla\cdot Q_T + \rho\mathbf{g}\cdot\mathbf{U}, \tag{2.3}$$

where $\mathbf{U}(t,x,y,z) = (u,v,w)^T$ is the velocity, ρ is the mass density, p is the fluid pressure, σ is the viscosity stress and $\mathbf{g} = (0,0,-g)^T$ represents the gravity forces. The internal specific (i.e. per mass unit) energy is denoted by e, the temperature by T. The heat flux Q_T obeys the Fourier law $Q_T = -\lambda\nabla T$, hence the name "Navier-Stokes-Fourier", λ being the heat conductivity. The quantity ∇ denotes $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$. In the following, we will also use the notations $\mathbf{u}$ and $\nabla_{x,y}$, $\mathbf{u}(t,x,y,z) = (u,v)^T$ is the horizontal velocity and $\nabla_{x,y}$ corresponds to the projection of ∇ on the horizontal plane i.e. $\nabla_{x,y} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$. The square norm of the velocity vector is $|\mathbf{U}|^2 = u^2 + v^2 + w^2$.

We consider a free surface flow (see Figure 2.1), therefore we assume

$$z_b(x,y) \leq z \leq \eta(t,x,y) := h(t,x,y) + z_b(x,y)$$

with $z_b(x,y)$ the bottom elevation and $h(t,x,y)$ the water depth.

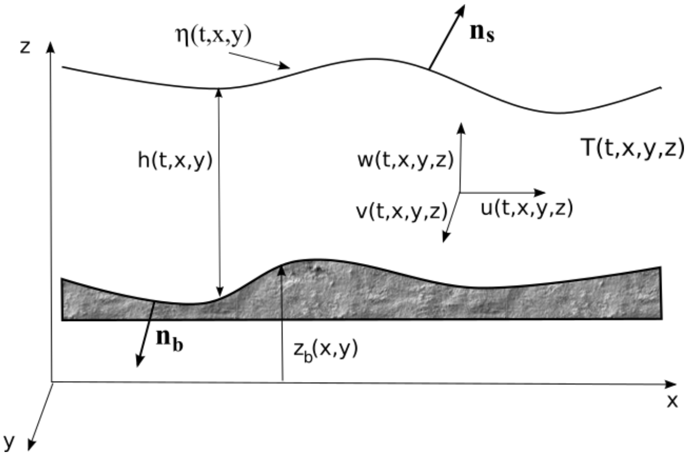

FIG. 2.1. *Flow domain with water height $h(t,x,y)$, free surface $\eta(t,x,y)$ and bottom $z_b(x,y)$.*

Notice that, because of the free surface and the gravitational forces, the term $\rho \mathbf{g}.\mathbf{U} = -\rho g w$ in the right-hand side of (2.3) prevents this equation from being directly a local energy conservation law. Nevertheless one can write (see Remark 2.1)

$$\rho g w = \frac{\partial(\rho g z)}{\partial t} + \nabla.(\rho g z \mathbf{U}),$$

which leads to a conservative version of (2.3). In the sequel, we will mainly work with the energy balance written under the form (2.3).

Regarding constitutive equations, we assume that the fluid is Newtonian i.e. the viscous part of the Cauchy stress depends linearly on the velocity gradient. Hence the stress tensor $\mathbf{\Sigma}$ is given by

$$\mathbf{\Sigma} \equiv -p\mathbf{1} + \sigma = -p\mathbf{1} + \zeta \nabla \cdot \mathbf{U}\mathbf{1} + 2\mu D(\mathbf{U}), \tag{2.4}$$

where μ is the viscosity coefficient, ζ is the second viscosity and $D(\mathbf{U}) = (\nabla \mathbf{U} + (\nabla \mathbf{U})^T)/2$. Further on we shall use the notation $A:B$ for the scalar product between two square matrices A, B, and $|A|^2 = A:A$.

Among the thermodynamic variables ρ, p, T, e, only two of them are independent. This implies in particular that we have a state law under the form

$$f(\rho,T,p) = 0. \tag{2.5}$$

The thermodynamic variables are linked by the identity

$$de = \frac{p}{\rho^2} d\rho + T ds, \tag{2.6}$$

where s is the specific entropy of the fluid. Classically, in order to have a good entropy structure one has to assume that $-s$ is a convex function of $1/\rho, e$. In Section 3.2 the case for which there is an additional thermodynamic variable S, the specific salinity, is described.

Energy equations can be deduced from the above equations. Multiplying (2.2) by $\mathbf{U}$ yields the kinetic energy equation

$$\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{U}|^2}{2}\right)+\nabla\cdot\Big(\big(\rho\frac{|\mathbf{U}|^2}{2}+p-\sigma\big)\mathbf{U}\Big)=p\nabla\cdot\mathbf{U}-\sigma:D(\mathbf{U})+\rho\mathbf{g}\cdot\mathbf{U}. \tag{2.7}$$

Subtracting (2.7) to (2.3) gives the equation for the internal energy

$$\frac{\partial\rho e}{\partial t}+\nabla\cdot(\rho e\mathbf{U})=-p\nabla\cdot\mathbf{U}+\sigma:D(\mathbf{U})-\nabla\cdot Q_T,$$

or equivalently

$$\rho\frac{De}{Dt}=-p\nabla\cdot\mathbf{U}+\sigma:D(\mathbf{U})-\nabla\cdot Q_T, \tag{2.8}$$

with the classical notation $D/Dt\equiv\partial/\partial t+\mathbf{U}\cdot\nabla$. We can write the continuity Equation (2.1) as

$$\rho\frac{D\rho}{Dt}+\rho^2\nabla\cdot\mathbf{U}=0. \tag{2.9}$$

With the thermodynamic relation (2.6) one can write $ds=de/T-(p/T\rho^2)d\rho$, thus multiplying (2.8) by $1/T$ and (2.9) by $-p/T\rho^2$ we obtain

$$\rho\frac{Ds}{Dt}=\frac{1}{T}\sigma:D(\mathbf{U})-\frac{1}{T}\nabla\cdot Q_T.$$

This can be written also

$$\frac{\partial\rho s}{\partial t}+\nabla\cdot(\rho s\mathbf{U})=\frac{1}{T}\sigma:D(\mathbf{U})-\nabla\cdot\frac{Q_T}{T}-Q_T\cdot\frac{\nabla T}{T^2}, \tag{2.10}$$

which gives the increase with time of $\int\rho s$, the second principle of thermodynamics.

2.2. Boundary conditions.

2.2.1. Bottom and free surface.

Let $\mathbf{n}_b$ and $\mathbf{n}_s$ be the unit outward normals at the bottom and at the free surface, respectively, defined by (see Figure 2.1)

$$\mathbf{n}_b=\frac{1}{\sqrt{1+|\nabla_{x,y}z_b|^2}}\begin{pmatrix}\nabla_{x,y}z_b\\-1\end{pmatrix},\qquad \mathbf{n}_s=\frac{1}{\sqrt{1+|\nabla_{x,y}\eta|^2}}\begin{pmatrix}-\nabla_{x,y}\eta\\1\end{pmatrix}.$$

On the bottom we prescribe an impermeability condition

$$\mathbf{U}\cdot\mathbf{n}_b=0, \tag{2.11}$$

and a friction condition given, for example, by a Navier law

$$(\mathbf{\Sigma}\cdot\mathbf{n}_b)\cdot\mathbf{t}_i=-\kappa\mathbf{U}\cdot\mathbf{t}_i,\quad i=1,2, \tag{2.12}$$

with κ a Navier coefficient and $(\mathbf{t}_i,i=1,2)$ two tangential vectors. For some applications, we rather use more specific friction laws and the Equation (2.12) is then replaced by

$$(\Sigma\cdot\mathbf{n}_b)\cdot\mathbf{t}_i=-\kappa(h,\mathbf{U})\cdot\mathbf{t}_i,\quad i=1,2,$$

with $\kappa(h,\mathbf{U}).\mathbf{U}\geq 0$. On the free surface, we use the kinematic boundary condition

$$\frac{\partial \eta}{\partial t}+\mathbf{u}(t,x,y,\eta)\cdot\nabla_{x,y}\eta-w(t,x,y,\eta)=0, \tag{2.13}$$

and the dynamic boundary condition

$$\mathbf{\Sigma}\cdot\mathbf{n}_s=-p^a(t,x,y)\mathbf{n}_s+W(t,x,y)\mathbf{t}_W, \tag{2.14}$$

where $p^a(t,x,y)$, $W(t,x,y)$ are two given quantities, p^a (resp. W) mimics the effects of the atmospheric pressure (resp. the wind blowing at the free surface) and $\mathbf{t}_W$ is a unit horizontal vector given by the wind direction. Throughout the paper $p^a=cst$, $W=0$. For the temperature, Neumann or Dirichlet boundary conditions can be taken, see Subsection 2.2.3.

Remark 2.1. Computing the quantity

$$\int_{z_b}^{\eta} z\left(\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})\right)dz,$$

and using the boundary conditions (2.13), (2.11) one finds

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} g\rho z dz+\nabla_{x,y}\cdot\int_{z_b}^{\eta} g\rho z\mathbf{u}dz=\int_{z_b}^{\eta} g\rho w dz,$$

which is the integrated local conservation of gravitational potential energy.

2.2.2. Fluid boundaries and solid walls.

On solid walls we prescribe a non-penetration condition (neglecting viscosity)

$$\mathbf{U}\cdot\mathbf{n}=0,$$

$\mathbf{n}$ being the outward normal to the considered wall, together with appropriate conditions for the temperature.

In this paper we consider fluid boundaries on which we prescribe zero, one or two of the following conditions depending on the type of the flow (fluvial or torrential): water level $h+z_b(x,y)$ given, discharge $h\mathbf{U}\cdot\mathbf{n}$ given.

2.2.3. Boundary conditions for the temperature.

The heat flux in Equation (2.3) requires to define boundary conditions for the temperature. Moreover when the state law (2.5) will be precised, the definition of the temperature at each boundary will be mandatory. We can choose either Neumann or Dirichlet conditions namely at the bottom

$$\lambda\frac{\partial T}{\partial \mathbf{n}_b}=\phi_T|_{z_b}, \tag{2.15}$$

or

$$T_b=T_b^0, \tag{2.16}$$

and at the free surface

$$\lambda\frac{\partial T}{\partial \mathbf{n}_s}=\phi_T|_s, \tag{2.17}$$

or

$$T_s = T_s^0, \tag{2.18}$$

where $\phi_T|_{z_b}$, $\phi_T|_s$ are two given temperature fluxes and T_b^0, T_s^0 are two given temperatures.

The system is completed with some initial conditions

$$h(0,x,y) = h^0(x,y), \quad \rho(0,x,y) = \rho^0(x,y), \quad \mathbf{U}(0,x,y,z) = \mathbf{U}^0(x,y,z).$$

3. The low Mach limit

In this section, the low Mach limit of the compressible Navier-Stokes equations is performed in order to get an incompressible system. As already mentioned in the introduction, one of the motivations for this limit is that the density of the water varies very little with pressure variations, and removing acoustic waves from the model is advantageous from the computational point of view.

3.1. The Navier-Stokes-Fourier with temperature. Therefore, we now consider the state equation of the fluid (2.5) under the form

$$\tilde{f}\big(\rho, T, \varepsilon(p - p_{ref})\big) = 0, \tag{3.1}$$

where $\varepsilon \ll 1$ is a small parameter (related to the sound velocity, see Remark 3.1) and with p_{ref} a reference pressure constant in space and time. In other words, taking into account the implicit function theorem, this means to assume the particular form for the pressure

$$p = p_{ref} + \frac{p_0}{\varepsilon}, \tag{3.2}$$

where the law $p_0(\rho, T)$ has no stiff scale.

Remark 3.1. When writing Equation (3.1), we assume that the density of the water depends very weakly on the pressure, and this is true in practice. Water and seawater compressibility values at various temperatures and pressures are given in [12, 26]. A possible state law for seawater (involving the salinity S, which we will include later on in Section 3.2) is to write (3.1) as

$$\rho(S,T,p) = \frac{\rho(S,T,p_{ref})}{1 - \frac{p - p_{ref}}{K(S,T,p-p_{ref})}},$$

where in the fraction $(p - p_{ref})/K(S,T,p-p_{ref})$, the denominator is very large with respect to the numerator, so that this law could actually be written

$$\rho(S,T,p) = \frac{\rho(S,T,p_{ref})}{1 - \varepsilon(p - p_{ref})}. \tag{3.3}$$

This law was published in [27], where values of the density ρ at different pressures and constant S,T are also given. One can see that the density varies slowly with respect to the pressure. As reported in Appendix A of [28], for $S = 8$ PSU and $T = 10$°C, the water density for $p = 0$ (atmospheric pressure) is $\rho(8,10,0) = 1005.945659$ kg.m^{-3} while for $p = 10$ bar, the density is $\rho(8,10,10) = 1006.41797$ kg.m^{-3}. The sound velocity is defined by

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s,$$

implying after some computations involving relations (2.6) and (3.3) that

$$\varepsilon \approx \frac{1}{\rho(S,T,p_{ref})c^2}.$$

Since in sea water, $c \approx 1500$ m.s^{-1}, this justifies the assumption $\varepsilon \ll 1$ done in relation (3.1).

Remark 3.2. One could consider that the reference pressure p_{ref} varies in time, for instance, because of changes in the boundary conditions of the system - p_{ref} adapts to temperature fluxes and mass fluxes at the boundaries. Here, for the sake of simplicity, we consider that p_{ref} is constant in space and in time.

For the thermodynamic identity (2.6) to be compatible with (3.2), it is necessary to consider a rescaling for e and s, leading to the following rescaled thermodynamic identities

$$e + \frac{p_{ref}}{\rho} = \frac{e_0}{\varepsilon}, \qquad s = \frac{s_0}{\varepsilon}, \quad \text{with} \quad de_0 = \frac{p_0}{\rho^2} d\rho + T ds_0, \tag{3.4}$$

where e_0 and s_0 do not involve stiff scales. The low Mach limit is performed by letting ε go to 0 in the Navier-Stokes-Fourier system with the relations (3.2), (3.4). As p is the physical pressure, it has to remain finite. Therefore according to (3.2), at the limit we get $p_0(\rho,T) = 0$. This can be written

$$T = T^{eq}(\rho),$$

or equivalently $\rho = \rho(T^{eq})$. In all the remainder of the paper, the superscript ^{eq} (as in e_0^{eq}) is used for the quantities "at equilibrium", i.e. quantities in which ρ and T are constrained by the relation $p_0(\rho,T) = 0$.

One can define the specific enthalpy $H = e + p/\rho$, and the specific heat capacity at constant pressure $c_p = \left(\frac{\partial H}{\partial T}\right)_p$. Then with (3.2), (3.4) one has $H = H_0/\varepsilon$ with $H_0 = e_0 + p_0/\rho$, and $c_p = c_{p_0}/\varepsilon$ with $c_{p_0} = \left(\frac{\partial H_0}{\partial T}\right)_{p_0}$.

We have the following result.

Proposition 3.1. *The system*

$$\nabla \cdot \mathbf{U} = -\frac{\rho'(T^{eq})}{\rho^2 c_p} \nabla \cdot (\lambda \nabla T^{eq}), \tag{3.5}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{3.6}$$

$$\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g}, \tag{3.7}$$

with the relation $T = T^{eq}(\rho)$, $\rho'(T) = \frac{d\rho}{dT}$ *and where* p *is a Lagrange multiplier, is the formal limit of the system* (2.1)-(2.3), *with* (3.2), (3.4) *as* ε *goes to 0. The energy balance verified by* (3.5)-(3.7) *is*

$$\frac{\partial}{\partial t}\left(\rho \frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho \frac{e_0^{eq}}{\varepsilon}\right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho \frac{e_0^{eq}}{\varepsilon} + p - \sigma\right)\mathbf{U}\right)$$
$$= \nabla \cdot \left(\frac{\lambda_0}{\varepsilon} \nabla T^{eq}\right) + \rho \mathbf{g} \cdot \mathbf{U} + (p - p_{ref}) \nabla \cdot \mathbf{U} - \sigma : D(\mathbf{U}). \tag{3.8}$$

The energy balance (3.8) is expressed in the rescaled variables defined by (3.2), (3.4) in order to show clearly the order of magnitude of each term. The quantity λ_0 is defined by (3.11).

Proof. We first rewrite (2.1) under the form

$$\rho\frac{D\rho}{Dt} = -\rho^2\nabla\cdot\mathbf{U}. \tag{3.9}$$

When $\varepsilon\to 0$ we get $\rho=\rho(T^{eq})$, and we can multiply (3.9) by $dT^{eq}/d\rho$ to get an equation for the temperature

$$\rho\frac{DT^{eq}}{Dt} = -\rho^2\frac{dT^{eq}}{d\rho}\nabla\cdot\mathbf{U}. \tag{3.10}$$

In the sequel, we consider that for the heat conduction λ and the fluid viscosity μ, we are in the following asymptotic regime,

$$\lambda = \frac{\lambda_0}{\varepsilon}, \quad \text{and} \quad \mu\sim 1. \tag{3.11}$$

Notice that if λ or μ is smaller than these scales, then the corresponding terms in the incompressible system (3.5)-(3.8) will simply vanish. Hence, the computation of (2.8)$-p_{ref}/\rho^2\times$(2.9), multiplying by ε and taking the limit, we get according to (3.4)

$$\rho\frac{De_0^{eq}}{Dt} = \nabla\cdot(\lambda_0\nabla T^{eq}). \tag{3.12}$$

Writing then as above the rescaled enthalpy $H_0=e_0+p_0/\rho$, one gets that at equilibrium

$$H_0^{eq}=e_0^{eq},$$

and $H_0^{eq}=H_0^{eq}(T^{eq})$. One has then according to the definition of c_{p_0}

$$dH_0^{eq}=c_{p_0}dT^{eq}.$$

We thus get a second equation for the temperature

$$\rho c_{p_0}\frac{DT^{eq}}{Dt} = \nabla\cdot(\lambda_0\nabla T^{eq}). \tag{3.13}$$

Comparing (3.10) and (3.13) we obtain (3.5), where we have replaced the scaled quantities c_{p_0} and λ_0 by their physical values εc_p and $\varepsilon\lambda$ respectively. The pressure p in (3.7) can finally be interpreted as a Lagrange multiplier for the Equation (3.5). The momentum Equation (3.7) together with the mass Equation (3.9) gives again the kinetic energy equation (2.7). Adding it to (3.12) divided by ε and to trivial terms in p_{ref} finally gives the energy balance (3.8). ∎

Remark 3.3. At the limit, the thermodynamic identity (3.4) becomes

$$de_0^{eq}=T^{eq}ds_0^{eq}.$$

From (3.12) we obtain the equation for the evolution of the entropy

$$\frac{\partial}{\partial t}(\rho s_0^{eq})+\nabla\cdot(\rho s_0^{eq}\mathbf{U})-\frac{1}{T^{eq}}\nabla\cdot(\lambda_0\nabla T^{eq})=0.$$

Written in the conservative/dissipative form, this gives

$$\frac{\partial}{\partial t}(\rho s_0^{eq})+\nabla\cdot(\rho s_0^{eq}\mathbf{U})-\nabla\cdot\left(\lambda_0\frac{\nabla T^{eq}}{T^{eq}}\right)=\lambda_0\frac{|\nabla T^{eq}|^2}{(T^{eq})^2},$$

which shows that in accordance with the second law of thermodynamics, the total entropy $\int \rho s_0^{eq}$ can only increase.

The energy balance (3.8) exhibits discrepancies of the order ε with respect to the original energy balance (2.3). Namely, the terms $(p-p_{ref})\nabla\cdot\mathbf{U}-\sigma:D(\mathbf{U})$ are of size ε with respect to the leading order terms and they are not present in (2.3). In order to get an energy balance closer to the original compressible equations, we go one step further and some corrections of size ε are incorporated into the system. The strategy consists in rescaling the variable part of the pressure, and is very similar to what is classically obtained in the literature when the low-Mach limit of the Navier-Stokes equations with thermal conduction is taken, see for instance [19]. These corrections do not modify the order of accuracy of the approximation however, since other error terms of order ε are not corrected.

PROPOSITION 3.2. *The system*

$$\nabla\cdot\mathbf{U}=-\frac{\rho'(T^{eq})}{\rho^2 c_p}(\nabla\cdot(\lambda\nabla T^{eq})+\sigma:D(\mathbf{U})), \tag{3.14}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0, \tag{3.15}$$

$$\frac{\partial(\rho\mathbf{U})}{\partial t}+\nabla\cdot(\rho\mathbf{U}\otimes\mathbf{U})+\nabla p-\nabla\cdot\sigma=\rho\mathbf{g}, \tag{3.16}$$

with T a given function of ρ

$$T=T^{eq}(\rho), \tag{3.17}$$

and where p is a Lagrange multiplier, is an approximation of order ε of the formal limit (3.5)-(3.7) *of the system* (2.1)-(2.3) *with* (3.2), (3.4). *The system* (3.14)-(3.16) *satisfies the energy balance equation*

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{U}|^2}{2}-p_{ref}+\rho\frac{e_0^c}{\varepsilon}\right)+\nabla\cdot\left((\rho\frac{|\mathbf{U}|^2}{2}-p_{ref}+\rho\frac{e_0^c}{\varepsilon}+p-\sigma)\mathbf{U}\right)\\ =&\nabla\cdot\left(\frac{\lambda_0}{\varepsilon}\nabla T^{eq}\right)+\rho\mathbf{g}\cdot\mathbf{U},\end{aligned} \tag{3.18}$$

where e_0^c is an independent (rescaled) corrected energy variable such that $e_0^c-e_0^{eq}=O(\varepsilon)$.

Proof. The proof is similar to that of Proposition 3.1. At the limit, the variables are constrained by the relation (3.17). To obtain an energy balance close to (2.3), we define an independent corrected energy variable e_0^c ($e_0^c\neq e_0^{eq}$) satisfying

$$\rho\frac{De_0^c}{Dt}=\nabla\cdot(\lambda_0\nabla T^{eq})-\varepsilon(p-p_{ref})\nabla\cdot\mathbf{U}+\varepsilon\sigma:D(\mathbf{U}). \tag{3.19}$$

With this correction, the energy balance is now (3.18). The internal energy at equilibrium e_0^{eq} satisfies (3.19) but without the term $\varepsilon(p-p_{ref})\nabla\cdot\mathbf{U}$, which implies that

$e_0^c - e_0^{eq}$ is of order ε. Recalling that $H_0^{eq} = e_0^{eq}$, we get the equation for the enthalpy at equilibrium H_0^{eq}

$$\rho \frac{DH_0^{eq}}{Dt} = \nabla\cdot(\lambda_0 \nabla T^{eq}) + \varepsilon\sigma : D(\mathbf{U}). \tag{3.20}$$

Equation (3.20) appears as a correction of (3.12). Notice that if the correction $-\varepsilon(p - p_{ref})\nabla\cdot\mathbf{U}$ were incorporated, the model obtained would be even more accurate, but it would contain derivatives of p, which makes the equations much more difficult to handle since p is a Lagrange multiplier. Here we restrict ourselves to the correction $\varepsilon\sigma : D(\mathbf{U})$, which is viscous dissipation, to keep the model simple.

From Equation (3.20), we get a corrected equation on the temperature

$$\rho c_{p_0} \frac{DT^{eq}}{Dt} = \nabla\cdot(\lambda_0 \nabla T^{eq}) + \varepsilon\sigma : D(\mathbf{U}).$$

Consequently, comparing with (3.10), we get the equation on $\nabla\cdot\mathbf{U}$ (in the rescaled variables)

$$\nabla\cdot\mathbf{U} = -\frac{\rho'(T^{eq})}{\rho^2 c_{p_0}}\left(\nabla\cdot(\lambda_0\nabla T^{eq}) + \varepsilon\sigma : D(\mathbf{U})\right),$$

which is (3.14). In physical variables, the equation for the temperature can also be written

$$\rho c_p \frac{DT^{eq}}{Dt} = -\nabla\cdot Q_T + \sigma : D(\mathbf{U}). \tag{3.21}$$

□

Remark 3.4. The entropy equation for the model (3.14)-(3.16) is

$$\frac{\partial}{\partial t}(\rho s_0^{eq}) + \nabla\cdot(\rho s_0^{eq}\mathbf{U}) - \nabla\cdot\left(\lambda_0 \frac{\nabla T^{eq}}{T^{eq}}\right) = \lambda_0 \frac{|\nabla T^{eq}|^2}{(T^{eq})^2} + \frac{\varepsilon}{T^{eq}}\sigma : D(\mathbf{U}).$$

As $\sigma : D(\mathbf{U}) \geq 0$, we obtain again that the total entropy $\int \rho s_0^{eq}$ can only increase. Here, there is no discrepancy with the original entropy equation (2.10).

Remark 3.5. In models (3.5)-(3.7) and (3.14)-(3.16), the temperature T^{eq} is no longer an independent variable of the system. It is recovered by inverting the state law $\rho = \rho(T^{eq})$.

Remark 3.6. In order to obtain the model (3.14)-(3.16) as the low-Mach limit of the Navier-Stokes equations with thermal conduction, the limit is usually performed by expanding the variables of the system in power series of the Mach number. The method we have used here is different since as we have given ourselves only a generic state law, the Mach number does not appear explicitly in the equations.

Remark 3.7. Even if a gas and water behave very differently, an example of equation of state to which the previous asymptotics can be applied is the stiffened gas law [29]

$$p = (\gamma - 1)\rho e - \gamma p_\infty, \qquad \frac{c_p}{\gamma} T = e - \frac{p_\infty}{\rho},$$

with the constraint $e - p_\infty/\rho > 0$, where $\gamma > 1$, $p_\infty > 0$, $c_p > 0$ are constants. Here the entropy is given by $s = \frac{c_p}{\gamma}\log(T/\rho^{\gamma-1})$. Then the scaling assumptions (3.2), (3.4) are satisfied when

$$p_\infty = -p_{ref} + \frac{p_{\infty 0}}{\varepsilon}, \qquad c_p = \frac{c_{p_0}}{\varepsilon},$$

with $p_{\infty 0}$ and c_{p_0} constants independent of ε. At equilibrium we get the relation $\rho T^{eq} = \frac{\gamma}{\gamma-1}\frac{p_{\infty 0}}{c_{p_0}}$, thus T^{eq} is inversely proportional to ρ.

Remark 3.8. In [24] the authors focused on the following 2D (x,z) model

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0, \tag{3.22}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u w}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z}, \tag{3.23}$$

$$\frac{\partial p}{\partial z} = -\rho g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}, \tag{3.24}$$

$$\rho = \rho(T), \tag{3.25}$$

$$\frac{\partial \rho T}{\partial t} + \frac{\partial \rho u T}{\partial x} + \frac{\partial \rho w T}{\partial z} = \frac{\lambda}{c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\lambda}{c_p}\frac{\partial^2 T}{\partial z^2}. \tag{3.26}$$

Rewriting Equation (3.26) in the non-conservative form, we get

$$\rho\frac{\partial T}{\partial t} + \rho u\frac{\partial T}{\partial x} + \rho w\frac{\partial T}{\partial z} = \frac{\lambda}{c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

Multiplying this equation by $\rho'(T)/\rho$ gives an equation for ρ

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = \frac{\rho'(T)}{\rho}\frac{\lambda}{c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right). \tag{3.27}$$

Finally, subtracting (3.22) to (3.27) and rearranging the terms gives a compatibility condition similar to (3.5) for a constant λ

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{\rho'(T)}{\rho^2 c_p}\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

Indeed (3.26) is similar to (3.13). This shows that the model (3.22)-(3.26) corresponds to the model (3.5)-(3.7) with a hydrostatic assumption, see paragraph 4.

Because of the stability inherited from the energy balance (3.18), that is consistent with (2.3), in the sequel we consider the system (3.14)-(3.16) instead of (3.5)-(3.7).

3.2. The Navier-Stokes-Fourier system with salinity. We now consider the situation where the fluid density depends on the temperature T and on another internal variable, the specific salinity S. This is the case of sea water. The compressible Navier-Stokes-Fourier system with temperature and salinity can be written

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{U}) = 0, \tag{3.28}$$

$$\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla\cdot(\rho \mathbf{U}\otimes\mathbf{U}) + \nabla p - \nabla\cdot\sigma = \rho \mathbf{g}, \tag{3.29}$$

$$\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{U}|^2}{2} + \rho e\right) + \nabla\cdot\left(\left(\rho\frac{|\mathbf{U}|^2}{2} + \rho e + p - \sigma\right)\mathbf{U}\right) = -\nabla\cdot\mathbf{F}^T + \rho\mathbf{g}\cdot\mathbf{U}, \tag{3.30}$$

$$\frac{\partial (\rho S)}{\partial t} + \nabla\cdot(\rho S \mathbf{U}) = -\nabla\cdot\mathbf{F}^S, \tag{3.31}$$

the properties and the expressions for the quantities $\mathbf{F}^T$ and $\mathbf{F}^S$ being discussed hereafter, see especially Equations (3.40) and (3.41). The local mass and momentum conservation equations are identical to (2.1) and (2.2), whereas the energy equation is slightly modified: the heat flux is now $\mathbf{F}^T$. The conservation Equation (3.31) on the mass fraction of chlorides S can also be written as

$$\rho\frac{DS}{Dt}=-\nabla\cdot\mathbf{F}^S, \tag{3.32}$$

with $\mathbf{F}^S$ the salt flux. According to [13], the molecular fluxes of heat and salt $\mathbf{F}^T$ and $\mathbf{F}^S$ are expressed in terms of the thermodynamic Onsager forces related to the entropy Equation (3.38) below,

$$\mathbf{F}^S=A\nabla\left(\frac{-\mu_S}{T}\right)+B\nabla\left(\frac{1}{T}\right), \tag{3.33}$$

$$\mathbf{F}^T=B\nabla\left(\frac{-\mu_S}{T}\right)+C\nabla\left(\frac{1}{T}\right), \tag{3.34}$$

where A, B and C are three independent coefficients to be discussed later, and μ_S is the chemical potential of seawater. The particular symmetric form of (3.33), (3.34) is related to Onsager's reciprocity principle and the dissipation of entropy, see (3.39) below. The state equation of the fluid is

$$f(\rho,T,S,p)=0,$$

and the thermodynamic identity now reads

$$de=\frac{p}{\rho^2}d\rho+Tds+\mu_S dS. \tag{3.35}$$

As in classical gas dynamics, a simple assumption that ensures the hyperbolic structure of the model is that $-s$ is a convex function of $1/\rho,e,S$. From (3.30) and using (3.29), (3.28) we get the equation on the internal energy

$$\rho\frac{De}{Dt}=-p\nabla\cdot\mathbf{U}+\sigma:D(\mathbf{U})-\nabla\cdot\mathbf{F}^T. \tag{3.36}$$

Let us explain how the formulas (3.33), (3.34) lead to the second law of thermodynamics. The equation for the entropy is obtained using the thermodynamic identity (3.35) combined with the mass, internal energy and salinity Equations (2.9), (3.36), (3.32),

$$\rho\left(\frac{\partial s}{\partial t}+\mathbf{U}\cdot\nabla s\right)=\frac{1}{T}\left(\sigma:D(\mathbf{U})-\nabla\cdot\mathbf{F}^T+\mu_S\nabla\cdot\mathbf{F}^S\right), \tag{3.37}$$

that can be written under conservative/dissipative form

$$\begin{aligned}&\frac{\partial\rho s}{\partial t}+\nabla\cdot(\rho s\mathbf{U})\\ &=\frac{1}{T}\sigma:D(\mathbf{U})-\nabla\cdot\left(\frac{1}{T}\mathbf{F}^T-\frac{\mu_S}{T}\mathbf{F}^S\right)+\mathbf{F}^T\cdot\nabla\left(\frac{1}{T}\right)-\mathbf{F}^S\cdot\nabla\left(\frac{\mu_S}{T}\right).\end{aligned} \tag{3.38}$$

Substituting the expressions (3.33), (3.34) in the right-hand side of (3.38), we obtain the following quadratic form for the nonconservative terms in the right-hand side of (3.38),

$$\mathbf{F}^T\cdot\nabla\left(\frac{1}{T}\right)-\mathbf{F}^S\cdot\nabla\left(\frac{\mu_S}{T}\right)=C\left|\nabla\left(\frac{1}{T}\right)\right|^2-2B\nabla\left(\frac{1}{T}\right)\cdot\nabla\left(\frac{\mu_S}{T}\right)+A\left|\nabla\left(\frac{\mu_S}{T}\right)\right|^2. \tag{3.39}$$

For this quadratic form to be nonnegative (which is required for entropy dissipation), the three constraints are $A>0$, $C>0$ and $AC>B^2$. With these constraints the expressions (3.33), (3.34) of $\mathbf{F}^S$ and $\mathbf{F}^T$ can be written in terms of the gradients of the salinity S, temperature T and pressure p (as in [13], equations (B.26) and (B.27)) by writing $\mu_S=\mu_S(T,S,p)$ and assuming $\partial_S\mu_S>0$,

$$\mathbf{F}^S=-\rho k^S\left(\nabla S+\frac{\partial_p\mu_S}{\partial_S\mu_S}\nabla p\right)-\left(\frac{\rho k^S T}{\partial_S\mu_S}\partial_T\left(\frac{\mu_S}{T}\right)+\frac{B}{T^2}\right)\nabla T, \tag{3.40}$$

$$\mathbf{F}^T=-\rho c_p k^T\nabla T+\frac{B\partial_S\mu_S}{\rho k^S T}\mathbf{F}^S, \tag{3.41}$$

where $k^T>0$ and $k^S>0$ are the thermal and molecular diffusivities of salt, related to A, B, C by

$$A=\frac{\rho k^S T}{\partial_S\mu_S},\qquad C=\rho c_p k^T T^2+\frac{B^2}{A}. \tag{3.42}$$

The free model coefficients are thus now k^S, k^T and B. Note that $\mathbf{F}^T$ in (3.41) is written as a gradient of T (as in the case where the temperature is the only tracer), plus another term, due to the presence of salt. Using (3.42) and (3.33), the quadratic form (3.39) can be rewritten

$$\begin{aligned}&C\left|\nabla\left(\frac{1}{T}\right)\right|^2-2B\nabla\left(\frac{1}{T}\right)\cdot\nabla\left(\frac{\mu_S}{T}\right)+A\left|\nabla\left(\frac{\mu_S}{T}\right)\right|^2\\ &=\rho c_p k^T T^2\left|\nabla\frac{1}{T}\right|^2+\frac{1}{A}\left|\mathbf{F}^S\right|^2,\end{aligned} \tag{3.43}$$

which shows that it is indeed nonnegative. Thus with (3.38) the total entropy $\int\rho s$ can only increase, in accordance with the second principle of thermodynamics.

We now perform the low Mach limit as in Section 3. We introduce the state equation of the fluid under the form

$$\tilde{f}\big(\rho,T,S,\varepsilon(p-p_{ref})\big)=0, \tag{3.44}$$

with $\varepsilon\ll 1$. Taking into account the thermodynamic identity (3.35), p, e, s are thus rescaled as in (3.2), (3.4) and μ_S scales as $1/\varepsilon$, which yields

$$p=p_{ref}+\frac{p_0}{\varepsilon},\quad e+\frac{p_{ref}}{\rho}=\frac{e_0}{\varepsilon},\quad s=\frac{s_0}{\varepsilon},\quad \mu_S=\frac{\mu_{S0}}{\varepsilon}, \tag{3.45}$$

with $de_0=\frac{p_0}{\rho^2}d\rho+Tds_0+\mu_{S0}dS$. We still define the specific enthalpy $H=e+p/\rho$, and the specific heat capacity at constant pressure $c_p=\left(\frac{\partial H}{\partial T}\right)_{p,S}$. Then one has $H=H_0/\varepsilon$ with $H_0=e_0+p_0/\rho$, and $c_p=c_{p_0}/\varepsilon$ with $c_{p_0}=\left(\frac{\partial H_0}{\partial T}\right)_{p_0,S}$. As $\varepsilon\to 0$, the finiteness of p yields the equilibrium relation $p_0(\rho,T,S)=0$ or equivalently $T=T^{eq}(\rho,S)$.

Proposition 3.3. *The system*

$$\begin{aligned}\nabla\cdot\mathbf{U}=&\frac{1}{\rho^2 c_p}\left(\frac{\partial\rho}{\partial T^{eq}}\right)_S\left((T^{eq})^2\left(\frac{\partial(\mu_S/T^{eq})}{\partial T^{eq}}\right)_S\nabla\cdot\mathbf{F}^S+\nabla\cdot\mathbf{F}^T-\sigma:D(\mathbf{U})\right)\\ &+\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S,\end{aligned} \tag{3.46}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0, \tag{3.47}$$

$$\frac{\partial(\rho\mathbf{U})}{\partial t}+\nabla\cdot(\rho\mathbf{U}\otimes\mathbf{U})+\nabla p-\nabla\cdot\sigma=\rho\mathbf{g}, \tag{3.48}$$

$$\frac{\partial(\rho S)}{\partial t}+\nabla\cdot(\rho S\mathbf{U})=-\nabla\cdot\mathbf{F}^S, \tag{3.49}$$

$$\mathbf{F}^S=-\rho k^S\nabla S-\left(\frac{\rho k^S T}{\partial_S\mu_S}\partial_T\left(\frac{\mu_S}{T}\right)+\frac{B}{T^2}\right)\nabla T, \tag{3.50}$$

$$\mathbf{F}^T=-\rho c_p k^T\nabla T+\frac{B\partial_S\mu_S}{\rho k^S T}\mathbf{F}^S, \tag{3.51}$$

with $T=T^{eq}(\rho,S)$ and where p is a Lagrange multiplier, is an approximation of order ε of the formal limit of the system (3.28)-(3.31), (3.40), (3.41), (3.45).

Proof. At the limit there remain only two independent thermodynamic variables, and one can take T and S. We consider that $\mathbf{F}^S$ is bounded but $\mathbf{F}^T\sim 1/\varepsilon$. With (3.33), (3.34) this means that $A\sim\varepsilon$, $B\sim 1$, $C\sim 1/\varepsilon$, and using (3.42) that

$$k^S\sim 1,\quad c_p k^T\sim 1/\varepsilon. \tag{3.52}$$

At equilibrium the term in ∇p in (3.40) disappears in the expression of $\mathbf{F}^S$ since μ_S depends only weakly on p (this is a consequence of the scaling assumption (3.44)) and ∇p remains bounded, giving (3.50), (3.51). The mass conservation Equation (3.28), the momentum Equation (3.29) and the salinity Equation (3.31) are unchanged. Consider then the enthalpy $H=e+p/\rho=H_0/\varepsilon$. Taking (3.36)$-p_{ref}/\rho^2\times$(2.9), multiplying by ε and taking the limit, we get according to (3.45) and using the correction as in Section 3 the equation for the rescaled enthalpy at equilibrium $H_0^{eq}=e_0^{eq}$,

$$\rho\frac{DH_0^{eq}}{Dt}=-\varepsilon\nabla\cdot\mathbf{F}^T+\varepsilon\sigma:D(\mathbf{U}). \tag{3.53}$$

One can write at equilibrium

$$dH^{eq}=\left(\frac{\partial H^{eq}}{\partial T^{eq}}\right)_S dT^{eq}+\left(\frac{\partial H^{eq}}{\partial S}\right)_{T^{eq}}dS. \tag{3.54}$$

Combining (3.53) (written in the physical variables) with (3.54) gives

$$\rho\left(\frac{\partial H^{eq}}{\partial T^{eq}}\right)_S\frac{DT^{eq}}{Dt}-\left(\frac{\partial H^{eq}}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S+\nabla\cdot\mathbf{F}^T-\sigma:D(\mathbf{U})=0. \tag{3.55}$$

We have similarly for the density

$$d\rho=\left(\frac{\partial\rho}{\partial T^{eq}}\right)_S dT^{eq}+\left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}dS. \tag{3.56}$$

The quantities $\left(\frac{\partial\rho}{\partial T^{eq}}\right)_S$ and $\left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}$ are known from the state law of salted water, and using (3.9), (3.32) we deduce another equation on the temperature

$$-\rho^2\nabla\cdot\mathbf{U}=\left(\frac{\partial\rho}{\partial T^{eq}}\right)_S\rho\frac{DT^{eq}}{Dt}-\left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S. \tag{3.57}$$

Combining (3.55) with (3.57) gives an expression for $\rho^2\nabla\cdot\mathbf{U}$

$$\rho^2\nabla\cdot\mathbf{U}=\frac{\left(\frac{\partial\rho}{\partial T^{eq}}\right)_S}{\left(\frac{\partial H^{eq}}{\partial T^{eq}}\right)_S}\left(-\left(\frac{\partial H^{eq}}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S+\nabla\cdot\mathbf{F}^T-\sigma:D(\mathbf{U})\right)+\left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S, \tag{3.58}$$

that generalizes (3.14). We recall that by definition $c_p=\left(\frac{\partial H}{\partial T}\right)_{p,S}$, thus $c_p=c_{p_0}/\varepsilon$ and at equilibrium $c_{p_0}=\left(\frac{\partial H_0}{\partial T^{eq}}\right)_S$, or in physical variables

$$c_p=\left(\frac{\partial H^{eq}}{\partial T^{eq}}\right)_S, \tag{3.59}$$

which enables to express the denominator in (3.58). Note that we obtain (3.58) without using the thermodynamic identity (3.35), and without involving μ_S. Next in (3.58) it remains to express $\partial H^{eq}/\partial S$. One has at equilibrium (see [13], equations (A.11.1) and (A.11.2))

$$\left(\frac{\partial H^{eq}}{\partial S}\right)_{T^{eq}}=\mu_S-T^{eq}\frac{\partial\mu_S}{\partial T^{eq}}=-(T^{eq})^2\frac{\partial}{\partial T^{eq}}(\mu_S/T^{eq}), \tag{3.60}$$

thus finally (3.58) gives (3.46). The relation (3.60) can be deduced from the limit of the thermodynamic identity (3.45). At equilibrium we have

$$ds_0^{eq}=\frac{dH_0^{eq}}{T^{eq}}-\frac{\mu_{S0}}{T^{eq}}dS, \tag{3.61}$$

that can be interpreted as $dH^{eq}=T^{eq}ds^{eq}+\mu_S dS$, analogous to (3.35). We reformulate (3.61) as

$$d\left(s_0^{eq}-\frac{H_0^{eq}}{T^{eq}}\right)=\frac{H_0^{eq}}{(T^{eq})^2}dT^{eq}-\frac{\mu_{S0}}{T^{eq}}dS.$$

The left-hand side is an exact differential form, therefore we can write that the two cross derivatives with respect to T,S and S,T are equal. It yields

$$\frac{\partial}{\partial T^{eq}}\left(-\frac{\mu_{S0}}{T^{eq}}\right)=\frac{\partial}{\partial S}\left(\frac{H_0^{eq}}{(T^{eq})^2}\right),$$

which gives (3.60). □

REMARK 3.9. Because of (3.61), (3.53), the entropy Equation (3.38) is still valid for our model (3.46)-(3.51), and the quadratic form on the right-hand side takes the form (3.43).

REMARK 3.10. A criterion of well-posedness of our incompressible system (3.46)-(3.51) can be derived as follows. We write that the second-order terms in the coupled S and T^{eq} Equations (3.49), (3.55) give a diffusion matrix with positive eigenvalues. With (3.50), (3.51) we have at equilibrium (using (3.60))

$$\mathbf{F}^S=-\rho k^S\nabla S-E\nabla T, \qquad \text{with } E=\frac{\rho k^S T}{\partial_S\mu_S}\partial_T\left(\frac{\mu_S}{T}\right)+\frac{B}{T^2}, \tag{3.62}$$

$$
\begin{aligned}
\mathbf{F}^T-\left(\frac{\partial H^{eq}}{\partial S}\right)_T\mathbf{F}^S &= -\rho c_p k^T\nabla T+\left(\frac{B\partial_S\mu_S}{\rho k^S T}+T^2\partial_T(\mu_S/T)\right)\mathbf{F}^S\\
&= -\rho c_p k^T\nabla T+E\frac{T\partial_S\mu_S}{\rho k^S}\mathbf{F}^S. \qquad (3.63)
\end{aligned}
$$

The diffusion matrix of the system is thus, taking into account (3.59),

$$
\begin{pmatrix} \rho k^S & E\\ E\frac{T\partial_S\mu_S}{c_p} & E^2\frac{T\partial_S\mu_S}{\rho k^S c_p}+\rho k^T\end{pmatrix}.
$$

We obtain positive eigenvalues under the natural conditions also mentioned in [13]

$$
k^S>0,\quad k^T>0,\quad \partial_S\mu_S>0,\quad c_p>0. \qquad (3.64)
$$

REMARK 3.11. For the particular choice of B such that $E=0$ (the relation between E and B is (3.62)) we have a diagonal diffusion matrix in S, T. In this case we have particular formulas

$$
\mathbf{F}^S=-\rho k^S\nabla S, \qquad (3.65)
$$

and from (3.63), (3.60)

$$
\mathbf{F}^T+T^2\partial_T\left(\frac{\mu_S}{T}\right)\mathbf{F}^S=-\rho c_p k^T\nabla T. \qquad (3.66)
$$

We deduce that

$$
\nabla\cdot\mathbf{F}^T+T^2\partial_T\left(\frac{\mu_S}{T}\right)\nabla\cdot\mathbf{F}^S=-\nabla\cdot\left(\rho c_p k^T\nabla T\right)-\mathbf{F}^S\cdot\nabla\left(T^2\partial_T\left(\frac{\mu_S}{T}\right)\right), \qquad (3.67)
$$

that can be used in (3.46). Next we write (3.55) using (3.59), (3.60) and (3.67) to get

$$
\rho c_p\frac{DT}{Dt}=\nabla\cdot\left(\rho c_p k^T\nabla T\right)+\mathbf{F}^S\cdot\nabla\left(T^2\partial_T\left(\frac{\mu_S}{T}\right)\right)+\sigma:D(\mathbf{U}). \qquad (3.68)
$$

We can write

$$
\begin{aligned}
\nabla\left(T^2\partial_T\left(\frac{\mu_S}{T}\right)\right) &= \nabla\left(-\mu_S+T\partial_T\mu_S\right)\\
&= -\partial_T\mu_S\nabla T-\partial_S\mu_S\nabla S+\nabla\left(T\partial_T\mu_S\right)\\
&= -\partial_S\mu_S\nabla S+T\nabla\left(\partial_T\mu_S\right),
\end{aligned}
$$

thus we obtain

$$
\rho c_p\frac{DT}{Dt}=\nabla\cdot\left(\rho c_p k^T\nabla T\right)+T\mathbf{F}^S\cdot\nabla\left(\partial_T\mu_S\right)-\partial_S\mu_S\,\mathbf{F}^S\cdot\nabla S+\sigma:D(\mathbf{U}), \qquad (3.69)
$$

that generalizes (3.21). We therefore obtain the Equations (3.49) and (3.69) on S and T, with the value of $\mathbf{F}^S$ given by (3.65). With the conditions (3.64) we can see that T remains nonnegative and S verifies the maximum principle (i.e. it remains within its initial lower and upper bounds). Notice that with (3.59), (3.60), the model finally relies on the knowledge of $\rho^{eq}(T,S)$, $H^{eq}(T,S)$, $k^T(T,S)$, $k^S(T,S)$.

4. Hydrostatic approximation of the Navier-Stokes-Fourier system

We consider the incompressible Navier-Stokes-Fourier system (3.14)-(3.16) derived in paragraph 3, for the sake of simplicity we consider that the density only depends on the temperature $\rho=\rho(T)$, and we neglect the salinity S. For the sake of lightness, the exponent ^{eq} is dropped in this part and in the rest of the present document. The system is completed with the boundary conditions (2.11)-(2.14).

The hydrostatic assumption consists in neglecting the vertical acceleration of the fluid

$$\rho\left(\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial vw}{\partial y}+\frac{\partial w^2}{\partial z}\right)\approx 0, \tag{4.1}$$

where we recall that $\mathbf{U}=(\mathbf{u},w)$ with $\mathbf{u}=(u,v)$. We refer to [30–32] for the analysis of hydrostatic models and for their asymptotic derivation [33–35].

4.1. The incompressible and hydrostatic Navier-Stokes-Fourier system.

Using the assumption (4.1) and the defintion (2.4), the hydrostatic approximation of the incompressible Navier-Stokes-Fourier system (3.14)-(3.16) therefore reads

$$\nabla\cdot\mathbf{U}=-\frac{\rho'(T)}{\rho^2 c_p}\Bigg(\nabla\cdot(\lambda\nabla T)+\zeta\big(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\big)^2 \\ +2\mu\Big(|D_{x,y}\mathbf{u}|^2+\frac{1}{2}|\nabla_{x,y}w+\partial_z\mathbf{u}|^2+(\partial_z w)^2\Big)\Bigg), \tag{4.2}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0, \tag{4.3}$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t}+\nabla_{x,y}\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\frac{\partial(\rho\mathbf{u}w)}{\partial z}+\nabla_{x,y}p \\ =\nabla_{x,y}\Big(\zeta\big(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\big)\Big)+\nabla_{x,y}\cdot\Big(2\mu D_{x,y}\mathbf{u}\Big)+\partial_z\Big(\mu\big(\nabla_{x,y}w+\partial_z\mathbf{u}\big)\Big), \tag{4.4}$$

$$\partial_z p=-\rho g+\partial_z\Big(\zeta\big(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\big)\Big)+\nabla_{x,y}\cdot\Big(\mu(\nabla_{x,y}w+\partial_z\mathbf{u})\Big)+\partial_z(2\mu\partial_z w), \tag{4.5}$$

with $T=T(\rho)$ a given function. According to Remark 3.2, p_{ref} is taken to be the atmospheric pressure p^a that is supposed constant.

The energy balance for the hydrostatic system (4.2)-(4.5) is

$$\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{u}|^2}{2}+\rho e^c\right)+\nabla\cdot\left(\left(\rho\frac{|\mathbf{u}|^2}{2}+\rho e^c+p-p^a\right)\mathbf{U}\right) \\ +\nabla_{x,y}\cdot\Big(-\zeta\big(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\big)\mathbf{u}-2\mu(D_{x,y}\mathbf{u})\mathbf{u}-\mu\big(\nabla_{x,y}w+\partial_z\mathbf{u}\big)w\Big) \\ +\partial_z\Big(-\zeta\big(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\big)w-\mu\big(\nabla_{x,y}w+\partial_z\mathbf{u}\big)\cdot\mathbf{u}-2\mu w\partial_z w\Big)=\nabla.(\lambda\nabla T)-\rho g w,$$

with the corrected internal energy e^c governed by

$$\frac{\partial}{\partial t}(\rho e^c)+\nabla\cdot(\rho e^c\mathbf{U})=-(p-p^a)\big(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\big)+\nabla\cdot(\lambda\nabla T) \\ +\zeta\big(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\big)^2+2\mu\Big(|D_{x,y}\mathbf{u}|^2+\frac{1}{2}|\nabla_{x,y}w+\partial_z\mathbf{u}|^2+(\partial_z w)^2\Big). \tag{4.6}$$

Notice that (4.6) is similar to (3.19) divided by ε. The energy $e(T)$ defined by $de/dT=c_p$ (e corresponds to e^{eq} in paragraph 3) satisfies the same Equation (4.6) except the term $-(p-p^a)\nabla\cdot\mathbf{U}$ in the right-hand side. It is obtained by multiplying (4.3) by $c_p dT/d\rho$ and by using (4.2).

In order to make the numerical approximation easier and without loss of applicability we propose a few usual simplifications for the viscous terms. First, let us notice that for water the second viscosity cannot be neglected compared to the dynamic viscosity, see [36]. But in practice, the chosen value for the coefficient μ (typically $\mu\approx 0.1$ kg.m^{-1}.s^{-1}) is much bigger than the value of the dynamic viscosity ($\approx 10^{-3}$ kg.m^{-1}.s^{-1}) in order to model the turbulence effects. Hence hereafter, we can neglect the second viscosity i.e. we consider $\zeta=0$.

Notice that we have from Equation (4.2) $\nabla\cdot\mathbf{U}=\mathcal{O}\big((\lambda+\mu)\rho'/(\rho^2c_p)\big)$ and therefore the terms in $\mathcal{O}\big(\mu(\lambda+\mu)\rho'/(\rho^2c_p)\big)\ll 1$ appearing in Equations (4.2), (4.4) can be neglected. Likewise, neglecting the viscous terms in Equation (4.5) and in the third component of (2.14), an integration between z and η of (4.5) gives the relation

$$p=p^a+\int_z^\eta \rho g dz_1. \tag{4.7}$$

Hence we propose to replace the system (4.2)-(4.5) by the following simplified system with constant μ

$$\nabla\cdot\mathbf{U}=-\frac{\rho'(T)}{\rho^2c_p}\left(\nabla\cdot(\lambda\nabla T)+\mu|\nabla_{x,y}\mathbf{u}|^2+\mu\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2\right), \tag{4.8}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0, \tag{4.9}$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t}+\nabla_{x,y}\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\frac{\partial(\rho\mathbf{u}w)}{\partial z}+\nabla_{x,y}\int_z^\eta \rho g dz_1=\mu\nabla_{x,y}.\nabla_{x,y}\mathbf{u}+\mu\frac{\partial^2\mathbf{u}}{\partial z^2}. \tag{4.10}$$

The system (4.8)-(4.10) is completed with the energy balance

$$\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{u}|^2}{2}+\rho e\right)+\nabla_{x,y}\cdot\left(\mathbf{u}\left(\rho\frac{|\mathbf{u}|^2}{2}+\int_z^\eta \rho g dz_1+\rho e\right)-\mu\nabla\frac{|\mathbf{u}|^2}{2}\right)=\nabla.(\lambda\nabla T),$$

with e is the internal energy of the fluid governed by

$$\frac{\partial(\rho e)}{\partial t}+\nabla_{x,y}\cdot(\rho e\mathbf{u})=\Big(\int_z^\eta \rho g dz_1\Big)\nabla_{x,y}\cdot\mathbf{u}+\mu|\nabla_{x,y}\mathbf{u}|^2+\mu\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2+\nabla.(\lambda\nabla T).$$

Therefore the formulation of the rheology terms appearing in (4.10) is similar to what has been studied by some of the authors in [37].

For the system (4.8)-(4.10), the boundary conditions (2.11), (2.13) and (2.15)-(2.18) remain valid. Concerning the dynamical boundary conditions for Equation (4.10), the assumptions used above for the viscous terms imply that the boundary condition (2.12) becomes

$$\mu\sqrt{1+|\nabla_{x,y}z_b|^2}\,\frac{\partial\mathbf{u}}{\partial\mathbf{n}_b}=-\kappa\mathbf{u}, \tag{4.11}$$

with κ a Navier coefficient. Notice that up to terms in $\mathcal{O}(\mu|\nabla_{x,y}z_b|)$, relations (2.12) and (4.11) are similar. For some applications, one can choose $\kappa=\kappa(h,\mathbf{u}|_b)$.

Likewise, the modified expression of the viscous terms leads to a simplified version of the dynamic boundary condition at the free surface and Equation (2.14) becomes

$$\mu \frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{n}_s} = W(t,x,y)\mathbf{t}_W, \tag{4.12}$$

where $\tilde{\mathbf{u}} = (\mathbf{u},0)^T$. The third component of (2.14) corresponds to $p_s = p^a$ that has been used to obtain (4.7)

4.2. The hydrostatic Euler-Fourier system. Assuming that the viscous terms are small, the incompressible Navier-Stokes-Fourier system (3.14)-(3.16) becomes the incompressible Euler-Fourier system

$$\nabla\cdot\mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p}\nabla\cdot(\lambda\nabla T), \tag{4.13}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{U}) = 0, \tag{4.14}$$

$$\frac{\partial(\rho\mathbf{U})}{\partial t} + \nabla\cdot(\rho\mathbf{U}\otimes\mathbf{U}) + \nabla p = \rho\mathbf{g}. \tag{4.15}$$

Therefore, the hydrostatic approximation of the system (4.13)-(4.15) consists in the hydrostatic Euler-Fourier model

$$\nabla\cdot\mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p}\nabla\cdot(\lambda\nabla T), \tag{4.16}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{U}) = 0, \tag{4.17}$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla_{x,y}\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \frac{\partial(\rho\mathbf{u}w)}{\partial z} + \nabla_{x,y}\int_z^\eta \rho g dz = 0, \tag{4.18}$$

completed with the relation $\rho = \rho(T)$ and the boundary conditions (2.11), (2.13).

4.3. Remark on the Boussinesq assumption. In geophysical water flows, density variations are often considered as small and this allows justifying the Boussinesq assumption, which consists in considering the density variations only in the gravitational forces. More precisely, assuming

$$\rho = \rho(T) = \rho_0 + f(T),$$

with $f(T) \ll \rho_0$, this leads to writing the incompressible hydrostatic Euler system (4.16)-(4.18) under the form

$$\nabla\cdot\mathbf{U} = 0, \tag{4.19}$$

$$\rho_0 c_p\left(\frac{\partial T}{\partial t} + \mathbf{U}\cdot\nabla T\right) = \nabla\cdot(\lambda\nabla T), \tag{4.20}$$

$$\rho_0\left(\frac{\partial\mathbf{u}}{\partial t} + \nabla_{x,y}\cdot(\mathbf{u}\otimes\mathbf{u}) + \frac{\partial(\mathbf{u}w)}{\partial z}\right) + \nabla_{x,y}\int_z^\eta \rho g dz = 0. \tag{4.21}$$

Notice that whereas in (4.16), the divergence of the velocity field equals the expansion due to the temperature effects, the Boussinesq assumption implies the divergence-free condition (4.19).

The Boussinesq assumption is valid in various regimes [11,38] but

- it does not ensure a conservation of the kinetic energy since $\rho_0 \frac{|\mathbf{u}|^2}{2}$ is conserved instead of $\rho\frac{|\mathbf{u}|^2}{2}$,
- for long-time phenomena (sloshing, wave propagation,...) significant differences appear when the Boussinesq assumption is made, see [24, paragraph 6.2].

In this work, the Boussinesq assumption is not done and some remarks about its validity are given in the forthcoming paper [25], see also [11, 38].

5. Conclusion

In this paper we have derived, using a low-Mach approximation, an incompressible model for variable density flows. This model is obtained from compressible Navier-Stokes equations by assuming that the density depends weakly on the pressure, thus involving a small parameter ε, that can eventually be identified as the Mach number. As $\varepsilon \to 0$, the pressure dependency is lost, while the dependency of the density on temperature and eventually salinity remains. It leads to an "incompressible" model where the divergence of the velocity is equal to source terms involving temperature and salinity fluxes. In order to obtain a more accurate model in terms of energy balance, a correction of order ε is incorporated, which is indeed the viscous dissipation. The resulting model does not rely on the Boussinesq approximation. It is mass-conservative; expansion or contraction can be observed as a result of the variation of a tracer concentration. Our models generalize classical incompressible models to quite general state laws that can depend on salinity. They include entropy consistency via Onsager's principles.

REFERENCES

[1] S. Rahmstorf, *Thermohaline circulation: The current climate*, Nature, 421:699, 2003. 1

[2] J.V. Boussinesq, *Théorie analytique de la chaleur mise en harmonie avec la thermodynamique et avec la théorie mécanique de la lumière*, Paris: Gathier-Villars, 2, 1903. 1

[3] E.A. Spiegel and G. Veronis, *On the Boussinesq approximation for a compressible fluid*, Astrophys. J., 131:442–447, 1960. 1

[4] Y. Lu, *Including non-Boussinesq effects in Boussinesq ocean circulation models*, J. Phys. Oceanogr., 31(6):1616–1622, 2000. 1

[5] F. Auclair, L. Bordois, Y. Dossmann, T. Duhaut, A. Paci, C. Ulses, and C. Nguyen, *A non-hydrostatic non-Boussinesq algorithm for free-surface ocean modelling*, Ocean Model., 132:12–19, 2018. 1

[6] ICON-ESM software, Home page, https://www.mpimet.mpg.de/en/science/models/icon-esm/. 1

[7] NEMO software, Home page, https://www.nemo-ocean.eu. 1

[8] POM software, Home page, http://www.ccpo.odu.edu/POMWEB/. 1

[9] N. Soontiens, M. Stastna, and M.L. Waite, *Trapped internal waves over topography: Non-Boussinesq effects, symmetry breaking and downstream recovery jumps*, Phys. Fluids, 25(6):066602, 2013. 1

[10] G.L. Mellor and T. Ezer, *Sea level variations induced by heating and cooling: An evaluation of the Boussinesq approximation in ocean models*, J. Geophys. Res., 100(C10):20565–20577, 1995. 1

[11] R.-J. Greatbatch, Y. Lu, and Y. Cai, *Relaxing the Boussinesq approximation in ocean circulation models*, J. Atmos. Ocean. Technol., 18(11):1911–1923, 2001. 1, 4.3

[12] J. Safarov, F. Millero, R. Feistel, A. Heintz, and E. Hassel, *Thermodynamic properties of standard seawater: extensions to high temperatures and pressures*, Ocean Sci., 5:235–246, 2009. 1, 3.1

[13] UNESCO, *Thermodynamic equation of seawater: Calculation and use of thermodynamic properties*, Programme and Meeting Document, 2010. 1, 3.2, 3.2, 3.2, 3.10

[14] R.X. Huang, X. Jin, and X. Zhang, *An oceanic general circulation model in pressure coordinates*, Adv. Atmos. Sci., 18:1-22, 2001. 1

[15] T.T. Song and T.Y. Hou, *Parametric vertical coordinate formulation for multiscale, Boussinesq, and non-Boussinesq ocean modeling*, Ocean Model., 11:298–332, 2006. 1

[16] Software MITgcm, http://mitgcm.org. 1

[17] S.M. Griffies, C. Böning, F.O. Bryan, E.P. Chassignet, R. Gerdes, H. Hasumi, A. Hirst, A.-M.

Treguier, and D. Webb, *Developments in ocean climate modelling*, Ocean Model., 2(3):123–192, 2000. 1

[18] CROCO software, Home page, https://www.croco-ocean.org. 1

[19] S. Paolucci, *On the filtering of sound from the Navier-Stokes equations*, Technical Report 82-8257, Sandia National Laboratories, 1982. 1, 3.1

[20] U. Kadri and M. Stiassnie, *Generation of an acoustic-gravity wave by two gravity waves, and their subsequent mutual interaction*, J. Fluid Mech., 735:R6, 2013. 1

[21] T. Alazard, *Low Mach number limit of the full Navier-Stokes equations*, Arch. Ration. Mech. Anal., 180(1):1–73, 2006. 1

[22] E. Feireisl and A. Novotný, *The low Mach number limit for the full Navier–Stokes–Fourier system*, Arch. Ration. Mech. Anal., 186:77–107, 2007. 1

[23] F. Bouchut, C. Chalons, and S. Guisset, *An entropy satisfying two-speed relaxation system for the barotropic Euler equations. Application to the numerical approximation of low Mach number flows*, Numer. Math., 145:35–76, 2020. 1

[24] E. Audusse, M.-O. Bristeau, M. Pelanti, and J. Sainte-Marie, *Approximation of the hydrostatic Navier-Stokes system for density stratified flows by a multilayer model. Kinetic interpretation and numerical validation*, J. Comput. Phys., 230:3453–3478, 2011. 1, 3.8, 4.3

[25] L. Boittin, F. Bouchut, M.-O. Bristeau, A. Mangeney, J. Sainte-Marie, and F. Souillé, *The Navier-Stokes system with temperature and salinity for free surface flows - Part II: Numerical scheme and validation*, preprint, 2020. 1, 4.3

[26] R. Fine and F. Millero, *Compressibility of water as a function of temperature and pressure*, J. Chem. Phys., 59(10):5529–5536, 1973. 3.1

[27] UNESCO, *Tenth report of the joint panel on oceanographic tables and standards*, Technical report, UNESCO Technical Papers in Marine Science, 1981. 3.1

[28] S.R. Massel, *Internal Gravity Waves in the Shallow Seas*, Springer, 2015. 3.1

[29] F. Harlow and A. Amsden, *Fluid dynamics*, Technical Report LA-4700, Los Alamos National Laboratory, 1971. 3.7

[30] Y. Brenier, *Homogeneous hydrostatic flows with convex velocity profiles*, Nonlinearity, 12(3):495–512, 1999. 4

[31] E. Grenier, *On the derivation of homogeneous hydrostatic equations*, ESAIM Math. Model. Numer. Anal., 33(5):965–970, 1999. 4

[32] N. Masmoudi and T. Wong, *On the H^s theory of hydrostatic Euler equations*, Arch. Ration. Mech. Anal., 204(1):231–271, 2012. 4

[33] E. Audusse, *A multilayer Saint-Venant model: Derivation and numerical validation*, Discrete Contin. Dyn. Syst. Ser. B, 5(2):189–214, 2005. 4

[34] S. Ferrari and F. Saleri, *A new two-dimensional shallow water model including pressure effects and slow varying bottom topography*, ESAIM Math. Model. Numer. Anal., 38(2):211–234, 2004. 4

[35] F. Marche, *Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects*, Eur. J. Mech. B/Fluids, 26:49–63, 2007. 4

[36] A. Dukhin and P. Goetz, *Bulk viscosity and compressibility measurement using acoustic spectroscopy*, J. Chem. Phys., 130:124519, 2009. 4.1

[37] S. Allgeyer, M.-O. Bristeau, D. Froger, R. Hamouda, V. Jauzein, A. Mangeney, J. Sainte-Marie, F. Souillé, and M. Vallée, *Numerical approximation of the 3D hydrostatic Navier-Stokes system with free surface*, ESAIM Math. Model. Numer. Anal., 53(6):1981–2024, 2019. 4.1

[38] T.J. McDougall, R.J. Greatbatch, and Y. Lu, *On conservation equations in oceanography: How accurate are Boussinesq ocean models?* J. Phys. Oceanogr., 32(5):1574–1584, 2002. 4.3