AN ENERGY-CONSISTENT DEPTH-AVERAGED EULER SYSTEM: DERIVATION AND PROPERTIES

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(Communicated by Benoit Perthame)

Abstract. In this paper, we present an original derivation process of a non-hydrostatic shallow water-type model which aims at approximating the incompressible Euler and Navier-Stokes systems with free surface. The closure relations are obtained by a minimal energy constraint instead of an asymptotic expansion. The model slightly differs from the well-known Green-Naghdi model and is confronted with stationary and analytical solutions of the Euler system corresponding to rotational flows. At the end of the paper, we give time-dependent analytical solutions for the Euler system that are also analytical solutions for the proposed model but that are not solutions of the Green-Naghdi model. We also give and compare analytical solutions of the two non-hydrostatic shallow water models.

1. **Introduction.** Despite the progress in the analysis and numerical approximation of the incompressible Euler and Navier-Stokes equations with free surface, there exists a demand for models of reduced complexity such as shallow water type models to represent gravity driven geophysical flows. In particular, the accurate description of the topography or bathymetry that play a key role in landslide dynamics or ocean wave propagation, requires simplified models to reduce the associated high computational cost.

Non-linear shallow water equations model the dynamics of a shallow, rotating layer of homogeneous incompressible fluid and are typically used to describe vertically averaged flows in two or three dimensional domains in terms of horizontal velocity and depth variations. The classical Saint-Venant system [3] with viscosity and friction [17, 18, 32] is particularly well-suited for the study and numerical

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simulations of a large class of geophysical phenomena such as rivers, lava flows, ice sheets, coastal domains, oceans or even run-off or avalanches when being modified with adapted source terms [6, 7, 30]. But the Saint-Venant system is built on the hydrostatic assumption consisting in neglecting the vertical acceleration of the fluid. This assumption is valid for a large class of geophysical flows but is restrictive in various situations where the dispersive effects – such as those occuring in wave propagation – cannot be neglected. As an example, neglecting the vertical acceleration in granular flows or landslides leads to significantly overestimate the initial flow velocity [31, 28], with strong implication for hazard assessment.

The modeling of the non-hydrostatic effects for shallow water flows does not raise insuperable difficulties [20, 12, 4, 36, 37, 10] but the analysis [1, 26] of the resulting models and their discretization become tough. The assumption of potential flows is often used to derive dispersive models and an extensive literature exists concerning these models. The most important contributions have been proposed by Lannes and co-authors [5, 13, 22, 1, 2], see also [16].

The non-hydrostatic model presented in this paper is not based on the irrotational assumption, on the other hand it is not derived using an asymptotic expansion of the incompressible Navier-Stokes or Euler based on the classical shallow water assumptions. Even if such an asymptotic expansion approach is natural, it leads to difficulties for the approximation of the non-hydrostatic pressure terms.

To overcome these problems, we propose a strategy for the model derivation that is widely used in the kinetic framework to obtain kinetic descriptions e.g. of conservations laws [25, 38]. The required closure relations to obtain a depth-averaged model approximating the Euler or Navier-Stokes system satisfy an energy-based optimality criterion. As a consequence, the proposed model slightly differs from existing models especially the well-known Green-Naghdi model [20, 26]. It consists in a set of first order partial differential equations and compared to the Green-Naghdi model, the contribution of the non-hydrostatic pressure terms differs from a scaling coefficient. Illustrating these differences, we give time-dependent analytical solutions for the Euler system that are also analytical solutions for the proposed model but that are not solution of the Green-Naghdi model.

The discretization of the proposed model is not in the scope of this paper, we only notice that numerical techniques have been recently proposed for the approximation of non-hydrostatic models but their properties (numerical cost/robustness) are not fully satisfactory [13, 9, 23] for practical uses especially in 2d with unstructured meshes. Since this model has the structure of a conservation law with additional terms and only contains first order derivatives, we hope that it can be discretized more easily using finite volume techniques.

The paper is organized as follows. In Section 2, we recall the incompressible Navier-Stokes equations with free surface with the associated boundary conditions and we deduce the Euler system. In Section 3 we derive the proposed non-hydrostatic model. Some of its properties are investigated in Section 4 and confrontations with analytical solutions are given in Section 5.

2. The Navier-Stokes and Euler systems.
In this section, we present the Navier-Stokes and Euler systems with their associated boundary conditions.

2.1. The Navier-Stokes equations.
The Navier-Stokes equations restricted to two dimensions have the following general formulation

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0, & \text{(1a)} \\[2ex] \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} + \dfrac{\partial p}{\partial x} = \dfrac{\partial \Sigma_{xx}}{\partial x} + \dfrac{\partial \Sigma_{xz}}{\partial z}, & \text{(1b)} \\[2ex] \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} + \dfrac{\partial p}{\partial z} = -g + \dfrac{\partial \Sigma_{zx}}{\partial x} + \dfrac{\partial \Sigma_{zz}}{\partial z}, & \text{(1c)} \end{cases}$$

where the z axis represents the vertical direction. We consider this system for

$$t > t_0, \quad x \in \mathbb{R}, \quad z_b(x,t) \le z \le \eta(x,t),$$

where $\eta(x,t)$ represents the free surface elevation, $\mathbf{u} = (u,w)^T$ the horizontal and vertical velocities. The water height is $H = \eta - z_b$, see Fig. 2.1. We consider that the bathymetry z_b can vary with respect to abscissa x and also with respect to time t. The chosen form of the viscosity stress tensor is symmetric

$$\Sigma_{xx} = 2\mu\frac{\partial u}{\partial x}, \qquad \Sigma_{xz} = \mu\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big),$$
$$\Sigma_{zz} = 2\mu\frac{\partial w}{\partial z}, \qquad \Sigma_{zx} = \mu\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big),$$

with μ the viscosity that is supposed constant. For a more general form of the viscosity tensor, see Ref. [15, 25]. We define the total stress tensor Σ_T

$$\Sigma_T = -pI_d + \Sigma.$$

As in Ref. [18], we introduce the indicator function for the fluid region

$$\varphi(x,z,t) = \begin{cases} 1 & \text{for } (x,z) \in \Omega = \{(x,z)\,|\,z_b \le z \le \eta\}, \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

The fluid region is advected by the flow, which can be expressed, thanks to the incompressibility condition, by the relation

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} = 0. \qquad (3)$$

The solution φ of this equation takes the values 0 and 1 only but it needs not be of the form (2) at all times. The analysis below is limited to the conditions where this
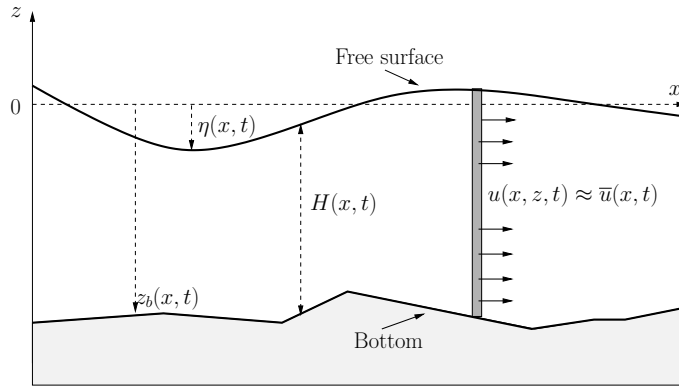

FIGURE 1. Notations: water height $H(x,t)$, free surface $\eta(x,t)$ and bottom $z_b(x,t)$.

form is preserved. For a more complete presentation of the Navier-Stokes system and its closure, the reader can refer to [27].

Remark 1. Notice that in the fluid domain, Eq. (3) reduces to the divergence free condition whereas across the upper and lower boundaries it gives the kinematic boundary conditions defined in the following.

2.2. **Boundary conditions.** The system (1) is completed with boundary conditions. The outward and upward unit normals to the free surface $\mathbf{n}_s$ and to the bottom $\mathbf{n}_b$ are given by

$$\mathbf{n}_s = \frac{1}{\sqrt{1 + \left(\frac{\partial \eta}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial \eta}{\partial x} \\ 1 \end{pmatrix}, \quad \mathbf{n}_b = \frac{1}{\sqrt{1 + \left(\frac{\partial z_b}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial z_b}{\partial x} \\ 1 \end{pmatrix}.$$

2.2.1. *At the free surface.* Classically at the free surface we have the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} - w_s = 0, \tag{4}$$

where the subscript s denotes the value of the considered quantity at the free surface. The dynamical condition at the free surface takes into account the equilibrium with the atmospheric pressure. Considering the air viscosity is negligible, the continuity of stresses at the free boundary imposes

$$\Sigma_T \mathbf{n}_s = -p^a(x, \eta(x, t), t) \mathbf{n}_s, \tag{5}$$

where $p^a = p^a(x, z, t)$ is a given function corresponding to the atmospheric pressure.

2.2.2. *At the bottom.* Since we consider that the bottom can vary with respect to time t, the kinematic boundary condition is

$$\frac{\partial z_b}{\partial t} + u_b \frac{\partial z_b}{\partial x} - w_b = 0, \tag{6}$$

where the subscript b denotes the value of the considered quantity at the bottom and $(x, t) \mapsto z_b(x, t)$ is a given function. Notice that Eq. (6) reduces to a classical no-penetration condition when z_b does not depend on time t.

For the stresses at the bottom we consider a friction law under the form

$$\Sigma_T \mathbf{n}_b - (\mathbf{n}_b . \Sigma_T \mathbf{n}_b) \mathbf{n}_b = \kappa \mathbf{v}_b, \tag{7}$$

with $\mathbf{v}_b = \mathbf{u}_b - (0, \frac{\partial z_b}{\partial t})^T$ the relative velocity between the water and the bottom and κ is a positive friction coefficient. Let $\mathbf{t}_b$ satisfies $(\mathbf{t}_b)^t \mathbf{n}_b = 0$ then after multiplication by $\mathbf{n}_b$, Eq. (7) leads to

$$(\mathbf{v}_b)^t \mathbf{n}_b = 0,$$

that is equivalent to Eq. (4). Similarly multiplying Eq. (7) by $\mathbf{t}_b$ gives

$$(\mathbf{t}_b)^t \Sigma_T \mathbf{n}_b = \kappa (\mathbf{v}_b)^t \mathbf{t}_b = \kappa \left(1 + \left(\frac{\partial z_b}{\partial x}\right)^2\right) u_b. \tag{8}$$

2.3. **Energy balance.** We recall the fondamental stability property related to the fact that the Navier-Stokes system admits an energy

$$E = E(z; u, w) = \frac{u^2 + w^2}{2} + gz, \tag{9}$$

leading to the following equation

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} (E + p^a) \, dz + \frac{\partial}{\partial x} \int_{z_b}^{\eta} \left[u(E + p) - \mu \left(2u \frac{\partial u}{\partial x} + w \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right) \right] dz =$$

$$- 2\mu \int \left[\left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] dz \tag{10}$$

$$+ H \frac{\partial p^a}{\partial t} + (p|_b - p^a) \frac{\partial z_b}{\partial t} - \kappa u_b.$$

2.4. **The Euler system.** Neglecting the viscous effects, we consider the Euler equations written in a conservative form

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \dfrac{\partial \varphi u}{\partial x} + \dfrac{\partial \varphi w}{\partial z} = 0, & \text{(11a)} \\[2mm] \dfrac{\partial \varphi u}{\partial t} + \dfrac{\partial \varphi u^2}{\partial x} + \dfrac{\partial \varphi uw}{\partial z} + \varphi \dfrac{\partial p}{\partial x} = 0, & \text{(11b)} \\[2mm] \dfrac{\partial \varphi w}{\partial t} + \dfrac{\partial \varphi uw}{\partial x} + \dfrac{\partial \varphi w^2}{\partial z} + \varphi \dfrac{\partial p}{\partial z} = -\varphi g, & \text{(11c)} \end{cases}$$

with φ defined by (2). The energy equation writes

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} (E + p^a) \, dz + \frac{\partial}{\partial x} \int_{z_b}^{\eta} u(E + p) = H \frac{\partial p^a}{\partial t} + (p|_b - p^a) \frac{\partial z_b}{\partial t}, \tag{12}$$

with E defined by (9). This system is completed with the boundary conditions (4),(5) and (6). In our case, (5) reduces to

$$p|_s = p^a. \tag{13}$$

For the sake of simplicity, in the following we neglect the variations of the atmospheric pressure p^a i.e. $p^a = p_0^a$ with $p_0^a = 0$.

2.5. **Non negativity of the pressure.** We also suppose in each point of the fluid region – including at the bottom – we have

$$p - p^a \geq 0.$$

The analysis below and especially the kinetic interpretation is restricted to this situation. Notice that in the case of hydrostatic Euler equations since we have

$$p - p^a = g(\eta - z),$$

this assumption reduces to the non-negativity of the water height H.

3. **Depth-averaged solutions of the Euler and Navier-Stokes systems.** In this section we take the vertical average of the Euler system and study the necessary closure relations for this system.

Let us denote $\langle f \rangle$ the average along the vertical axis, the so-called *depth-average*, of the quantity $f = f(z)$ i.e.

$$\langle f \rangle(x, t) = \int_{\mathbb{R}} f(x, z, t) \, dz. \tag{14}$$

During the derivation process of the model, we assume the bottom topography does not depend on time t, i.e.

$$\frac{\partial z_b}{\partial t} = 0.$$

The contribution of the time variations of the bottom topography is given in remark 3.

3.1. **Depth-averaging of the Euler solution.** The goal is to transpose the entropy-based moment closures proposed by Levermore in [24] for kinetic equations to our framework. In such a way, we obtain a nonperturbative derivation of shallow-water models which is justified by an entropy minimization process under constraint. The constraints concern the moments of the solution of the Euler equation, which are here the depth-averaged variables.

Taking into account the kinematic boundary conditions (4) and (6), the depth-averaged form of the Euler system (11) writes

$$\begin{cases} \dfrac{\partial}{\partial t}\langle \varphi \rangle + \dfrac{\partial}{\partial x}\langle \varphi u \rangle = 0, & \text{(15a)} \\[2mm] \dfrac{\partial}{\partial t}\langle \varphi u \rangle + \dfrac{\partial}{\partial x}\langle \varphi u^2 \rangle + \langle \varphi \dfrac{\partial p}{\partial x} \rangle = 0, & \text{(15b)} \\[2mm] \dfrac{\partial}{\partial t}\langle \varphi w \rangle + \dfrac{\partial}{\partial x}\langle \varphi u w \rangle + \langle \varphi \dfrac{\partial p}{\partial z} \rangle = -\langle \varphi g \rangle, & \text{(15c)} \\[2mm] \dfrac{\partial}{\partial t}\langle \varphi z \rangle + \dfrac{\partial}{\partial x}\langle \varphi z u \rangle = \langle \varphi w \rangle, & \text{(15d)} \end{cases}$$

where the last equation is a rewriting of

$$\langle \int_{z_b}^{z} \left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} \right) dz \rangle = \langle z \left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} \right) \rangle = 0,$$

using again the kinematic boundary conditions. Notice that using the definition (2), we have

$$\langle \varphi \rangle = H, \quad \text{and} \quad \langle \varphi z \rangle = \frac{\eta^2 - z_b^2}{2}. \tag{16}$$

Simple manipulations allow to obtain the system (15) from the Euler system (11),(4) and (6) e.g. for Eq. (15a), starting from (11a) we write

$$\langle \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} \rangle = 0,$$

and permuting the derivative with the integral using the Leibniz rule directly gives (15a).

We decompose the pressure p under the form

$$p = g(\eta - z) + p_{nh},$$

i.e. the sum of the hydrostatic and non-hydrostatic parts of the pressure. Hence, the system (15) becomes

$$\begin{cases} \dfrac{\partial}{\partial t}\langle \varphi \rangle + \dfrac{\partial}{\partial x}\langle \varphi u \rangle = 0, & \text{(17a)} \\[2mm] \dfrac{\partial}{\partial t}\langle \varphi u \rangle + \dfrac{\partial}{\partial x}\left(\langle \varphi u^2 \rangle + g\langle \varphi(\eta - z) \rangle + \langle \varphi p_{nh} \rangle \right) = -\left(g\langle \varphi \rangle + p_{nh}|_b \right)\dfrac{\partial z_b}{\partial x}, & \text{(17b)} \\[2mm] \dfrac{\partial}{\partial t}\langle \varphi w \rangle + \dfrac{\partial}{\partial x}\langle \varphi u w \rangle = p_{nh}|_b, & \text{(17c)} \\[2mm] \dfrac{\partial}{\partial t}\langle \varphi z \rangle + \dfrac{\partial}{\partial x}\langle \varphi z u \rangle = \langle \varphi w \rangle, & \text{(17d)} \end{cases}$$

where the boundary condition (13) has been used. The energy equation (12) gives

$$\frac{\partial}{\partial t}\langle \varphi E\rangle + \frac{\partial}{\partial x}\langle \varphi u(E+p)\rangle = 0, \tag{18}$$

where $E(z; u, w)$ is defined by (9).

Therefore the system (17) has four equations with four unknowns, namely $\langle \varphi \rangle$, $\langle \varphi u\rangle$, $\langle \varphi w\rangle$ and $\langle \varphi p_{nh}\rangle$ and closure relations are needed to define $\langle \varphi u^2\rangle$, $\langle \varphi uw\rangle$, $\langle \varphi zu\rangle$ and $p_{nh}|_b$.

If u', w' are defined as the deviations of u, w with respect to their depth-averages, then it comes

$$\varphi u = \varphi\frac{\langle \varphi u\rangle}{\langle \varphi\rangle} + \varphi u', \quad \varphi w = \varphi\frac{\langle \varphi w\rangle}{\langle \varphi\rangle} + \varphi w', \tag{19}$$

with $\langle \varphi u'\rangle = \langle \varphi w'\rangle = 0$. Following the moment closure proposed by Levermore [24], we study the minimization problem

$$\min_{u', w'}\langle \{\varphi E(z; u, w)\}\rangle. \tag{20}$$

The energy $E(z; u, w)$ being quadratic with respect to u we notice

$$\begin{aligned}
\langle \varphi u^2\rangle &= \frac{\langle \varphi u\rangle^2}{\langle \varphi\rangle} + 2\langle \varphi uu'\rangle + \langle \varphi(u')^2\rangle \\
&= \frac{\langle \varphi u\rangle^2}{\langle \varphi\rangle} + \langle \varphi(u')^2\rangle \\
&\geq \frac{\langle \varphi u\rangle^2}{\langle \varphi\rangle},
\end{aligned} \tag{21}$$

and similarly, we obtain

$$\langle \varphi w^2\rangle \geq \frac{\langle \varphi w\rangle^2}{\langle \varphi\rangle}. \tag{22}$$

Eqs. (21) and (22) mean that the solution of the minimization problem (20) is given by

$$\langle \varphi E\left(z; \frac{\langle \varphi u\rangle}{\langle \varphi\rangle}, \frac{\langle \varphi w\rangle}{\langle \varphi\rangle}\right)\rangle = \min_{u', w'}\langle \{\varphi E(z; u, w)\}\rangle, \tag{23}$$

and

$$\langle \varphi E\left(z; \frac{\langle \varphi u\rangle}{\langle \varphi\rangle}, \frac{\langle \varphi w\rangle}{\langle \varphi\rangle}\right)\rangle = \frac{\langle \varphi u\rangle^2 + \langle \varphi w\rangle^2}{2\langle \varphi\rangle} + g\langle \varphi z\rangle, \tag{24}$$

Since the only choice leading to equalities in relations (21) and (22) corresponds to

$$u = \frac{\langle \varphi u\rangle}{\langle \varphi\rangle}, \quad \text{and} \quad w = \frac{\langle \varphi w\rangle}{\langle \varphi\rangle}, \tag{25}$$

this allows to precise the closure relations associated to a minimal energy, namely

$$\left\{\begin{aligned}
\langle \varphi u^2\rangle &= \frac{\langle \varphi u\rangle^2}{\langle \varphi\rangle}, && \text{(26a)} \\
\langle \varphi uw\rangle &= \frac{\langle \varphi u\rangle\langle \varphi w\rangle}{\langle \varphi\rangle}, && \text{(26b)} \\
\langle \varphi zu\rangle &= \langle \varphi z\rangle\frac{\langle \varphi u\rangle}{\langle \varphi\rangle}. && \text{(26c)}
\end{aligned}\right.$$

Replacing (26) into Eqs. (17) leads to the system

$$
\begin{cases}
\dfrac{\partial}{\partial t}\langle\varphi\rangle + \dfrac{\partial}{\partial x}\langle\varphi u\rangle = 0, & \text{(27a)} \\[2ex]
\dfrac{\partial}{\partial t}\langle\varphi u\rangle + \dfrac{\partial}{\partial x}\left(\dfrac{\langle\varphi u\rangle^2}{\langle\varphi\rangle} + g\langle\varphi(\eta - z)\rangle + \langle\varphi p_{nh}\rangle\right) = \\[2ex]
\hspace{6cm} -\left(g\langle\varphi\rangle + p_{nh}|_b\right)\dfrac{\partial z_b}{\partial x}, & \text{(27b)} \\[2ex]
\dfrac{\partial}{\partial t}\langle\varphi w\rangle + \dfrac{\partial}{\partial x}\langle\varphi w\rangle\dfrac{\langle\varphi u\rangle}{\langle\varphi\rangle} = p_{nh}|_b, & \text{(27c)} \\[2ex]
\dfrac{\partial}{\partial t}\langle\varphi z\rangle + \dfrac{\partial}{\partial x}\langle\varphi z\rangle\dfrac{\langle\varphi u\rangle}{\langle\varphi\rangle} = \langle\varphi w\rangle, & \text{(27d)}
\end{cases}
$$

but it remains to find the closure relation for the non-hydrostatic pressure terms. As proved in the following proposition, the only possible choice is

$$
p_{nh}|_b = 2\frac{\langle\varphi p_{nh}\rangle}{\langle\varphi\rangle}. \tag{28}
$$

Proposition 1. *The solutions of the Euler system* (11)-(13),(6),(4) *satisfying the closure relations* (26),(28) *are also solutions of the system*

$$
\begin{cases}
\dfrac{\partial}{\partial t}\langle\varphi\rangle + \dfrac{\partial}{\partial x}\langle\varphi u\rangle = 0, & \text{(29a)} \\[2ex]
\dfrac{\partial}{\partial t}\langle\varphi u\rangle + \dfrac{\partial}{\partial x}\left(\dfrac{\langle\varphi u\rangle^2}{\langle\varphi\rangle} + g\langle\varphi(\eta - z)\rangle + \langle\varphi p_{nh}\rangle\right) = \\[2ex]
\hspace{5cm} -\left(g\langle\varphi\rangle + 2\dfrac{\langle\varphi p_{nh}\rangle}{\langle\varphi\rangle}\right)\dfrac{\partial z_b}{\partial x}, & \text{(29b)} \\[2ex]
\dfrac{\partial}{\partial t}\langle\varphi w\rangle + \dfrac{\partial}{\partial x}\langle\varphi w\rangle\dfrac{\langle\varphi u\rangle}{\langle\varphi\rangle} = 2\dfrac{\langle\varphi p_{nh}\rangle}{\langle\varphi\rangle}, & \text{(29c)} \\[2ex]
\dfrac{\partial}{\partial t}\langle\varphi z\rangle + \dfrac{\partial}{\partial x}\langle\varphi z\rangle\dfrac{\langle\varphi u\rangle}{\langle\varphi\rangle} = \langle\varphi w\rangle. & \text{(29d)}
\end{cases}
$$

This system is a depth-averaged approximation of the Euler system and admits – for smooth solutions – an energy balance under the form

$$
\begin{aligned}
\dfrac{\partial}{\partial t}&\langle\varphi E\left(z; \dfrac{\langle\varphi u\rangle}{\langle\varphi\rangle}, \dfrac{\langle\varphi w\rangle}{\langle\varphi\rangle}\right)\rangle \\
&+ \dfrac{\partial}{\partial x}\langle\dfrac{\langle\varphi u\rangle}{\langle\varphi\rangle}\left(\varphi E\left(z; \dfrac{\langle\varphi u\rangle}{\langle\varphi\rangle}, \dfrac{\langle\varphi w\rangle}{\langle\varphi\rangle}\right) + \langle\varphi p_{nh}\rangle\right)\rangle = 0.
\end{aligned} \tag{30}
$$

Remark 2. It is important to notice that whereas the solution H, u, w, p of the Euler system (11)-(13),(6),(4) also satisfies the system (17), only the solutions H, u, w, p of the Euler system (11)-(13),(6),(4) satisfying the closure relations (26),(28) are also solutions of the system (29)-(30). On the contrary, any solutions $\langle\varphi\rangle$, $\langle\varphi u\rangle$, $\langle\varphi w\rangle$ and $\langle p_{nh}\rangle$ of (29)-(29d) with $p_{nh}|_b$ defined by (28) are also solutions of (17)-(18).

Proof of prop. 1. Only the manipulations allowing to obtain (30) have to be detailed. More precisely, we have to prove that, in (27), the relation (28) is needed in order to obtain (30).

For that purpose, we multiply (27b) by $\frac{\langle \varphi u \rangle}{\langle \varphi \rangle}$ and we rewrite each of the obtained terms. For the terms also appearing in the Saint-Venant system i.e. corresponding to the hydrostatic part of the model, we easily obtain

$$\left(\frac{\partial}{\partial t} \langle \varphi u \rangle + \frac{\partial}{\partial x} \left(\frac{\langle \varphi u \rangle^2}{\langle \varphi \rangle} + g \langle \varphi (\eta - z) \rangle \right) + g \langle \varphi \rangle \frac{\partial z_b}{\partial x} \right) \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} = \\ \frac{\partial}{\partial t} \langle \varphi E \left(z; \frac{\langle \varphi u \rangle}{\langle \varphi \rangle}, 0 \right) \rangle + \frac{\partial}{\partial x} \langle \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \varphi E \left(z; \frac{\langle \varphi u \rangle}{\langle \varphi \rangle}, 0 \right) \rangle. \tag{31}$$

Multiplying (27c) by $\frac{\langle \varphi w \rangle}{\langle \varphi \rangle}$ and using (27a), we obtain the relation

$$\frac{\partial}{\partial t} \frac{\langle \varphi w \rangle^2}{2 \langle \varphi \rangle} + \frac{\partial}{\partial x} \frac{\langle \varphi u \rangle \langle \varphi w \rangle^2}{2 \langle \varphi \rangle^2} = \frac{\langle \varphi w \rangle}{\langle \varphi \rangle} \left. p_{nh} \right|_b. \tag{32}$$

And for the contribution of the non-hydrostatic pressure terms of Eq. (27b) over the energy balance, it comes

$$\begin{aligned} \left(\frac{\partial}{\partial x} \langle \varphi p_{nh} \rangle + \left. p_{nh} \right|_b \frac{\partial z_b}{\partial x} \right) \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} &= \frac{\partial}{\partial x} \frac{\langle \varphi p_{nh} \rangle \langle \varphi u \rangle}{\langle \varphi \rangle} - \langle \varphi p_{nh} \rangle \frac{\partial}{\partial x} \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \\ &\quad + \left. p_{nh} \right|_b \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \frac{\partial z_b}{\partial x} \\ &= \frac{\partial}{\partial x} \frac{\langle \varphi p_{nh} \rangle \langle \varphi u \rangle}{\langle \varphi \rangle} - \frac{\langle \varphi p_{nh} \rangle}{\langle \varphi \rangle} \frac{\partial \langle \varphi u \rangle}{\partial x} \\ &\quad + \frac{\langle \varphi p_{nh} \rangle \langle \varphi u \rangle}{\langle \varphi \rangle^2} \frac{\partial \langle \varphi \rangle}{\partial x} \\ &\quad + \left. p_{nh} \right|_b \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \frac{\partial z_b}{\partial x}. \end{aligned} \tag{33}$$

Since the identity

$$\langle \varphi z \rangle = \frac{\langle \varphi \rangle}{2} \left(\langle \varphi \rangle + 2 z_b \right),$$

holds, relation (27d) coupled with (27a) reduces to

$$\langle \varphi w \rangle = -\frac{\langle \varphi \rangle}{2} \frac{\partial \langle \varphi u \rangle}{\partial x} + \frac{\langle \varphi u \rangle}{2} \frac{\partial (\langle \varphi \rangle + 2 z_b)}{\partial x}, \tag{34}$$

and we can rewrite (33) under the form

$$\begin{aligned} \left(\frac{\partial}{\partial x} \langle \varphi p_{nh} \rangle + \left. p_{nh} \right|_b \frac{\partial z_b}{\partial x} \right) \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} &= \frac{\partial}{\partial x} \frac{\langle \varphi p_{nh} \rangle \langle \varphi u \rangle}{\langle \varphi \rangle} + 2 \frac{\langle \varphi p_{nh} \rangle}{\langle \varphi \rangle^2} \langle \varphi w \rangle \\ &\quad + \left(\left. p_{nh} \right|_b - 2 \frac{\langle \varphi p_{nh} \rangle}{\langle \varphi \rangle} \right) \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \frac{\partial z_b}{\partial x}. \end{aligned} \tag{35}$$

Adding (31),(32) and (35) gives

$$\begin{aligned} \frac{\partial}{\partial t} \langle \varphi E \left(z; \frac{\langle \varphi u \rangle}{\langle \varphi \rangle}, \frac{\langle \varphi w \rangle}{\langle \varphi \rangle} \right) \rangle &+ \frac{\partial}{\partial x} \langle \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \left(\varphi E \left(z; \frac{\langle \varphi u \rangle}{\langle \varphi \rangle}, \frac{\langle \varphi w \rangle}{\langle \varphi \rangle} \right) + \langle \varphi p_{nh} \rangle \right) \rangle \\ &= \left(\left. p_{nh} \right|_b - 2 \frac{\langle \varphi p_{nh} \rangle}{\langle \varphi \rangle} \right) \left(\frac{\langle \varphi w \rangle}{\langle \varphi \rangle} + \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \frac{\partial z_b}{\partial x} \right). \end{aligned} \tag{36}$$

Using (34) we have

$$\frac{\langle \varphi w \rangle}{\langle \varphi \rangle} + \frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \frac{\partial z_b}{\partial x} = -\frac{1}{2} \frac{\partial \langle \varphi u \rangle}{\partial x} + \frac{\langle \varphi u \rangle}{2 \langle \varphi \rangle} \frac{\partial \langle \varphi \rangle}{\partial x} = -\frac{\langle \varphi \rangle}{2} \frac{\partial}{\partial x} \left(\frac{\langle \varphi u \rangle}{\langle \varphi \rangle} \right),$$

and therefore the right hand side of (36) vanishes iff (28) holds that concludes the proof. □

3.2. **The proposed non-hydrostatic averaged model and other writings.** In the following, we no more handle variables corresponding to vertical means of the solution of the Euler equations (11). We adopt the notation $\overline{f} = f(x,t)$. By analogy with (29)-(30), we consider as non-hydrostatic averaged model the following system

$$
\begin{cases}
\dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\big(H\overline{u}\big) = 0, & \text{(37a)} \\[2ex]
\dfrac{\partial}{\partial t}(H\overline{u}) + \dfrac{\partial}{\partial x}\Big(H\overline{u}^2 + \dfrac{g}{2}H^2 + H\overline{p}_{nh}\Big) = -(gH + 2\overline{p}_{nh})\dfrac{\partial z_b}{\partial x}, & \text{(37b)} \\[2ex]
\dfrac{\partial}{\partial t}(H\overline{w}) + \dfrac{\partial}{\partial x}(H\overline{w}\,\overline{u}) = 2\overline{p}_{nh}, & \text{(37c)} \\[2ex]
\dfrac{\partial}{\partial t}\left(\dfrac{\eta^2 - z_b^2}{2}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{\eta^2 - z_b^2}{2}\overline{u}\right) = H\overline{w}. & \text{(37d)}
\end{cases}
$$

The smooth solutions H, $\overline{u}$, $\overline{w}$, $\overline{p}_{nh}$ of the system (37) also satisfies the energy balance

$$
\frac{\partial \overline{E}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{u}\big(\overline{E} + \frac{g}{2}H^2 + H\overline{p}_{nh}\big)\right) = 0, \tag{38}
$$

where

$$
\overline{E} = \frac{H(\overline{u}^2 + \overline{w}^2)}{2} + \frac{gH(\eta + z_b)}{2}. \tag{39}
$$

Notice that simple manipulations of Eqs. (37) lead to the relation

$$
H\overline{w} = -\frac{H}{2}\frac{\partial(H\overline{u})}{\partial x} + \frac{H\overline{u}}{2}\frac{\partial(H + 2z_b)}{\partial x}, \tag{40}
$$

corresponding to a shallow water expression of the divergence free condition.

The system (37)-(38) has been obtained by one of the authors in [39] but in the framework of asymptotic expansion. In this case, the justification of the closure relations is less obvious than using the energy-based optimality criterion (23).

Simple manipulations in the equations of (37) lead to different formulations of the model which are given in the two following corollaries.

Corollary 1. *The system* (37) *can be rewritten under the form*

$$
\begin{cases}
\dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\big(H\overline{u}\big) = 0, & \text{(41a)} \\[2ex]
\dfrac{\partial}{\partial t}(H\overline{u}) + \dfrac{\partial}{\partial x}\Big(H\overline{u}^2 + \dfrac{g}{2}H^2 + H\overline{p}_{nh}\Big) = -(gH + 2\overline{p}_{nh})\dfrac{\partial z_b}{\partial x}, & \text{(41b)} \\[2ex]
\dfrac{\partial}{\partial t}\big(H\overline{w}\big) + \dfrac{\partial}{\partial x}\big(H\overline{w}\,\overline{u}\big) = 2\overline{p}_{nh}, & \text{(41c)} \\[2ex]
H\overline{w} = -\dfrac{H}{2}\dfrac{\partial(H\overline{u})}{\partial x} + \dfrac{H\overline{u}}{2}\dfrac{\partial(H + 2z_b)}{\partial x}, & \text{(41d)}
\end{cases}
$$

and for smooth solutions Eq. (38) *remains valid.*

Corollary 2. *The system* (37) *can be rewritten under the form*

$$
\begin{cases}
\dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\big(H\overline{u}\big) = 0, & \text{(42a)} \\[2ex]
\dfrac{\partial}{\partial t}(H\overline{u}) + \dfrac{\partial}{\partial x}\left(H\overline{u}^2 + \dfrac{g}{2}H^2 + H\overline{p}_{nh}\right) = -(gH + 2\overline{p}_{nh})\dfrac{\partial z_b}{\partial x}, & \text{(42b)} \\[2ex]
\dfrac{\partial}{\partial t}\left(\dfrac{H^2}{2}\overline{w}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{H^2}{2}\overline{w}\,\overline{u}\right) = H\overline{p}_{nh} + H\overline{w}^2 - H\overline{u}\,\overline{w}\dfrac{\partial z_b}{\partial x}, & \text{(42c)} \\[2ex]
\dfrac{\partial}{\partial t}\left(\dfrac{H^2}{2}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{H^2}{2}\overline{u}\right) = H\overline{w} - H\overline{u}\dfrac{\partial z_b}{\partial x}, & \text{(42d)}
\end{cases}
$$

and for smooth solutions Eq. (38) *remains valid.*

Corollary 3. *The system* (37) *can be rewritten under the form*

$$
\begin{cases}
\dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\big(H\overline{u}\big) = 0, & \text{(43a)} \\[2ex]
\dfrac{\partial}{\partial t}(H\overline{u}) + \dfrac{\partial}{\partial x}(H\overline{u}^2) + \dfrac{\partial}{\partial x}\big(H\overline{p}\big) = -2\overline{p}\dfrac{\partial z_b}{\partial x}, & \text{(43b)} \\[2ex]
\dfrac{\partial}{\partial t}\left(\dfrac{\eta^2 - z_b^2}{2}\overline{w}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{\eta^2 - z_b^2}{2}\overline{w}\,\overline{u}\right) = (H + 2z_b)\overline{p} + H\overline{w}^2 - g\dfrac{\eta^2 - z_b^2}{2} & \text{(43c)} \\[2ex]
\dfrac{\partial}{\partial t}\left(\dfrac{\eta^2 - z_b^2}{2}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{\eta^2 - z_b^2}{2}\overline{u}\right) = H\overline{w}, & \text{(43d)}
\end{cases}
$$

and for smooth solutions Eq. (38) *remains valid.*

Proofs of corollaries 1, 2 and 3. Equation (42c) can be obtained multiplying Eq. (37c) by $\frac{H}{2}$ and using (40) and simple manipulations allow to obtain (42d) from (37d). Equation (43c) can be obtained multiplying Eq. (37c) by $\frac{H+2z_b}{2}$ and using (40). $\qquad\square$

Remark 3. When considering the bottom z_b can vary w.r.t. time t, the system (37) remains unchanged only the energy balance (38) is modified and becomes

$$
\frac{\partial \overline{E}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{u}\big(\overline{E} + \frac{g}{2}H^2 + H\overline{p}_{nh}\big)\right) = (gH + 2\overline{p}_{nh})\frac{\partial z_b}{\partial t}, \tag{44}
$$

with $\overline{E}$ defined by (39). Since $\overline{p}|_b = gH + 2\overline{p}_{nh}$, the contributions of the time variations of z_b in Eq. (44) are consistent with those appearing in (12).

3.3. **About asymptotic expansion.** For shallow water flows, the model derivation is often carried out using the shallow water assumption. Indeed, introducing the small parameter

$$
\varepsilon = \frac{h}{\lambda},
$$

where h and λ, two characteristic dimensions along the z and x axis respectively, an asymptotic expansion of the Euler or Navier-Stokes system leads to simplified averaged models such as the Saint-Venant system. As in [18, 15, 32, 39] and neglecting the viscous and friction effects, the shallow water assumption allows to justify the estimate

$$
u = \overline{u} + \mathcal{O}(\varepsilon^2), \tag{45}
$$

leading, using the divergence free condition, to

$$
w = -(z - z_b)\frac{\partial \overline{u}}{\partial x} + \overline{u}\frac{\partial z_b}{\partial x} + \mathcal{O}(\varepsilon^2). \tag{46}
$$

Inserting (45) and (46) in the momentum equation (11c) implies that the non-hydrostatic part of the pressure is linear in the variable z

$$\frac{\partial p_{nh}}{\partial z} = \alpha(x,t)(z - z_b) + \beta(x,t) + \mathcal{O}(\varepsilon^2).$$

Unfortunately, the preceding relation is not compatible with the closure relation for the pressure (28). And it is then necessary to add a scaling coefficient over the non-hydrostatic pressure terms in order to ensure the existence of an energy balance.

Notice that the energy balance obtained using the rescaled non-hydrostatic pressure terms differ from (30) and (38). The Green-Naghdi [20] can be derived using such an asymptotic expansion strategy.

3.4. **Comparison with Green-Naghdi model.** One of the most popular models for the description of long, dispersive water waves is the Green-Naghdi model. Several derivations of the Green-Naghdi model have been proposed in the litterature [20, 19, 40, 34]. For the mathematical justification of the model, the reader can refer to [1, 29] and for its numerical approximation to [23, 5, 13, 9].

Following [23] and with $z_b = cst$, the Green-Naghdi model reads

$$\begin{cases} \dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\left(H\overline{u}\right) = 0, & \text{(47a)} \\[3mm] \dfrac{\partial(H\overline{u})}{\partial t} + \dfrac{\partial}{\partial x}\left(H\overline{u}^2 + \dfrac{g}{2}H^2 + H\overline{p}_{gn}\right) = 0, & \text{(47b)} \end{cases}$$

with $\overline{p}_{gn} = \frac{1}{3}H\ddot{H}$ and the "dot" notation means the material derivative

$$\dot{H} = \frac{\partial H}{\partial t} + \overline{u}\frac{\partial H}{\partial x}. \tag{48}$$

When $z_b = cst$, the Green-Naghdi model and the non-hydrostatic model (37) are identical up to a multiplicative constant for the non-hydrostatic pressure. Indeed starting from the expression of $\overline{p}_{gn}$, the relations (47a) and (48) give

$$\begin{aligned} \overline{p}_{gn} &= \frac{1}{3}H\left(\frac{\partial \dot{H}}{\partial t} + \overline{u}\frac{\partial \dot{H}}{\partial x}\right) \\ &= \frac{1}{3}H\left(\frac{\partial}{\partial t}\left(-H\frac{\partial \overline{u}}{\partial x}\right) + \overline{u}\frac{\partial}{\partial x}\left(-H\frac{\partial \overline{u}}{\partial x}\right)\right). \end{aligned}$$

If we denote, as in (40)

$$\overline{w} = -\frac{H}{2}\frac{\partial \overline{u}}{\partial x}, \tag{49}$$

it comes

$$\overline{p}_{gn} = \frac{2}{3}H\left(\frac{\partial \overline{w}}{\partial t} + \overline{u}\frac{\partial \overline{w}}{\partial x}\right) = \frac{2}{3}\left(\frac{\partial}{\partial t}(H\overline{w}) + \frac{\partial}{\partial x}(H\overline{u}\overline{w})\right).$$

Therefore, the Green-Naghdi can also be written under the form

$$\begin{cases} \dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\left(H\overline{u}\right) = 0, & \text{(50a)} \\[3mm] \dfrac{\partial(H\overline{u})}{\partial t} + \dfrac{\partial}{\partial x}\left(H\overline{u}^2 + \dfrac{g}{2}H^2 + H\overline{p}_{gn}\right) = 0, & \text{(50b)} \\[3mm] \dfrac{\partial}{\partial t}(H\overline{w}) + \dfrac{\partial}{\partial x}(H\overline{u}\overline{w}) = \dfrac{3}{2}\overline{p}_{gn}, & \text{(50c)} \end{cases}$$

with the constraint (49) and completed, for smooth solutions, by the energy balance

$$\frac{\partial \overline{E}_{gn}}{\partial t} + \frac{\partial}{\partial x}\overline{u}\left(\overline{E}_{gn} + H\overline{p}_{gn}\right) = 0, \tag{51}$$

with

$$\overline{E}_{gn} = \frac{H}{2}\left(\overline{u}^2 + \frac{2}{3}\overline{w}^2\right) + \frac{g}{2}H^2. \tag{52}$$

The energy balance (51) illustrates the main difference between the Green-Nagdhi model and the proposed non-hydrostatic model (37)-(38). In the case of a flat bottom, (39) and (52) only differ by the coefficient $\frac{2}{3}$ in the vertical part of the kinetic energy.

To summarize, for flat bottom, choosing either $\gamma = 2$ or $\gamma = \frac{3}{2}$, the system

$$\begin{cases} \dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\left(H\overline{u}\right) = 0, & \text{(53a)} \\[2mm] \dfrac{\partial(H\overline{u})}{\partial t} + \dfrac{\partial}{\partial x}\left(H\overline{u}^2 + \dfrac{g}{2}H^2 + H\overline{p}\right) = 0, & \text{(53b)} \\[2mm] \dfrac{\partial}{\partial t}(H\overline{w}) + \dfrac{\partial}{\partial x}(H\overline{u}\overline{w}) = \gamma\overline{p}, & \text{(53c)} \\[2mm] \overline{w} = -\dfrac{H}{2}\dfrac{\partial \overline{u}}{\partial x}, & \text{(53d)} \end{cases}$$

corresponds to the depth-averaged system (37) or to the Green-Naghdi system (49)-(50), respectively. The system (53) is completed with the energy balance

$$\frac{\partial \overline{E}_{\gamma}}{\partial t} + \frac{\partial}{\partial x}\overline{u}\left(\overline{E}_{\gamma} + H\overline{p}\right) = 0, \tag{54}$$

with

$$\overline{E}_{\gamma} = \frac{H}{2}\left(\overline{u}^2 + \frac{1}{\gamma}\overline{w}^2\right) + \frac{g}{2}H^2. \tag{55}$$

Despite its similarities with the Green-Naghdi model, the non-hydrostatic model (37)-(38) has several advantages

- its derivation is more simple than the Green-Naghdi model (see [20, 19]),
- the topography source terms appear quite naturally (that is not the case for most of the versions available in the literature [11, 35]),
- the model formulation is written under the form of an advection-reaction set of PDE and does not contain high order derivatives.

A comparison between the solutions of the two non-hydrostatic models is obviously a key point but it requires a numerical scheme for their discretization that is not in the scope of this paper. We illustrate in paragraphs 5.1 and 5.2 the differences between the two non-hydrostatic models in the case of analytical solutions.

3.5. **Hydrostatic case.** The process used for the derivation of the non-hydrostatic model in paragraph 3.1 can also be used for the derivation of shallow water hydrostatic models.

The hydrostatic assumption in (11) that means that the contribution of the vertical acceleration in the pressure p can be neglected, leads to the classical model

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \dfrac{\partial \varphi u}{\partial x} + \dfrac{\partial \varphi w}{\partial z} = 0, & \text{(56a)} \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial uw}{\partial z} + \dfrac{\partial p}{\partial x} = 0, & \text{(56b)} \\[2ex] \dfrac{\partial p}{\partial z} = -g. & \text{(56c)} \end{cases}$$

This hydrostatic model – or some variants with horizontal and vertical viscosity or other specific terms – is often used in geophysical flows studies and it has been widely studied, let us mention some important contributions [8, 21, 33].

Starting from Eqs. (56), the shallow water assumption allows to derive the classical Saint-Venant system (see also [17, 18, 32])

$$\begin{cases} \dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\left(H\overline{u}\right) = 0, & \text{(57a)} \\[2ex] \dfrac{\partial(H\overline{u})}{\partial t} + \dfrac{\partial(H\overline{u}^2)}{\partial x} + \dfrac{g}{2}\dfrac{\partial H^2}{\partial x} = -gH\dfrac{\partial z_b}{\partial x}. & \text{(57b)} \end{cases}$$

The smooth solutions of (57) satisfy the energy equality

$$\dfrac{\partial E_h}{\partial t} + \dfrac{\partial}{\partial x}\left(\overline{u}\left(E_h + g\dfrac{H^2}{2}\right)\right) = 0, \tag{58}$$

with the energy

$$E_h = \dfrac{H\overline{u}^2}{2} + \dfrac{gH(\eta + z_b)}{2}. \tag{59}$$

Notice that (58),(59) corresponds to (9),(12) where the hydrostatic and shallow water assumptions are made.

3.6. A depth-averaged Navier-Stokes system.
In Section 3, we have started from the Euler system to obtain its depth-averaged version. In this section, we use the same process as in paragraphs 3 to obtain a depth-averaged Navier-Stokes system. And we have the following proposition

Proposition 2. *A depth-averaged version of the free surface Navier-Stokes system leads to the model*

$$\begin{cases} \dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}\left(H\overline{u}\right) = 0, & \text{(60a)} \\[2ex] \dfrac{\partial}{\partial t}(H\overline{u}) + \dfrac{\partial}{\partial x}\left(H\overline{u}^2 + \dfrac{g}{2}H^2 + H\overline{p}_{nh}\right) = \\[2ex] \qquad\qquad\qquad -(gH + 2\overline{p}_{nh})\dfrac{\partial z_b}{\partial x} + \dfrac{\partial}{\partial x}\left(2\mu H\dfrac{\partial \overline{u}}{\partial x}\right) - \kappa\overline{u}, & \text{(60b)} \\[2ex] \dfrac{\partial}{\partial t}(H\overline{w}) + \dfrac{\partial}{\partial x}(H\overline{wu}) = 2\overline{p}_{nh} + \dfrac{\partial}{\partial x}\left(\mu H\dfrac{\partial \overline{w}}{\partial x}\right), & \text{(60c)} \\[2ex] \dfrac{\partial}{\partial t}\left(\dfrac{\eta^2 - z_b^2}{2}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{\eta^2 - z_b^2}{2}\overline{u}\right) = H\overline{w}. & \text{(60d)} \end{cases}$$

Moreover the smooth solutions of (60) *satisfy the energy balance*

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial}{\partial x} \left(\overline{u} \left(\overline{E} + \frac{g}{2} H^2 + H \overline{p}_{nh} - 2\mu H \frac{\partial \overline{u}}{\partial x} \right) - \mu H \overline{w} \frac{\partial \overline{w}}{\partial x} \right)$$
$$= -\mu H \left(2 \left(\frac{\partial \overline{u}}{\partial x} \right)^2 + \left(\frac{\partial \overline{w}}{\partial x} \right)^2 \right) - \kappa \overline{u}^2, \tag{61}$$

with $\overline{E}$ *defined by* (39).

Proof of proposition 2. Compared to the derivation of the model (37)-(38), only the treatment of the viscous terms has to be precised and we have

$$\int \left(\frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xz}}{\partial z} \right) \varphi dz = \frac{\partial}{\partial x} \int 2\mu \frac{\partial u}{\partial x} \varphi dz - \kappa \overline{u},$$

where the boundary conditions (5),(7) have been used. And replacing u by $\overline{u}$ in the r.h.s. of the preceding relation gives the expression of the viscous term in (60b). Likewise, using (5),(7), we have

$$\int \left(\frac{\partial \Sigma_{zx}}{\partial x} + \frac{\partial \Sigma_{zz}}{\partial z} \right) \varphi dz = \frac{\partial}{\partial x} \int \mu \frac{\partial w}{\partial x} \varphi dz,$$

and replacing w by $\overline{w}$ gives the expression of the viscous term in (60c). Multiplying (60b) by $\overline{u}$ and (60c) by $\overline{w}$ and after simple manipulations, we obtain the relation (61) that completes the proof. $\square$

4. Some properties of the non-hydrostatic model.

4.1. **Expression for** $\overline{p}_{nh}$. Equation (37d) – that is equivalent to (40) – is not a dynamical equation but a constraint ensuring a shallow water version of the divergence free condition. And hence it plays a specific role in the non-hydrostatic model. We try to reformulate Eq. (40) in order to obtain an equation satisfied by the pressure $\overline{p}_{nh}$. The process used is similar to Chorin solenoidal decomposition of the velocity field [14] for Navier-Stokes equations.

The derivative w.r.t. time t of the shallow water form of the divergence free condition (40) gives

$$\frac{\partial (H\overline{w})}{\partial t} + \frac{H}{2} \frac{\partial^2 (H\overline{u})}{\partial x \partial t} - \frac{1}{2} \frac{\partial (H + 2z_b)}{\partial x} \frac{\partial (H\overline{u})}{\partial t} = -\frac{H\overline{u}}{2} \frac{\partial^2 (H\overline{u})}{\partial x^2} + \frac{1}{2} \left(\frac{\partial (H\overline{u})}{\partial x} \right)^2,$$

where relation (37a) has been used. Now substituting the expressions (37b),(37c) for

$$\frac{\partial (H\overline{u})}{\partial t}, \quad \text{and} \quad \frac{\partial (H\overline{w})}{\partial t},$$

in the previous relation gives

$$2\overline{p}_{nh} + \left(\frac{1}{2} \frac{\partial (H + 2z_b)}{\partial x} - \frac{H}{2} \frac{\partial}{\partial x} \right) \left(\frac{\partial (H\overline{p}_{nh})}{\partial x} + 2\overline{p}_{nh} \frac{\partial z_b}{\partial x} \right) = B, \tag{62}$$

with

$$B = \frac{1}{2} \left(\frac{\partial (H\overline{u})}{\partial x} \right)^2 - \frac{H\overline{u}}{2} \frac{\partial^2 (H\overline{u})}{\partial x^2} + \frac{H}{2} \left(\frac{\partial^2}{\partial x^2} \left(H\overline{u}^2 + \frac{g}{2} H^2 \right) + g \frac{\partial}{\partial x} \left(H \frac{\partial z_b}{\partial x} \right) \right)$$
$$+ \frac{\partial (H\overline{wu})}{\partial x} - \frac{1}{2} \frac{\partial (H + 2z_b)}{\partial x} \left(\frac{\partial}{\partial x} \left(H\overline{u}^2 + \frac{g}{2} H^2 \right) + gH \frac{\partial z_b}{\partial x} \right).$$

From (40), we get

$$\frac{\partial(H\overline{wu})}{\partial x} = -\frac{1}{2}\frac{\partial}{\partial x}\left(H\overline{u}\frac{\partial(H\overline{u})}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{H\overline{u}^2}{2}\frac{\partial(H+2z_b)}{\partial x}\right),$$

leading to

$$\begin{aligned}
B &= -H\overline{u}\frac{\partial^2(H\overline{u})}{\partial x^2} + \frac{H}{2}\left(\frac{\partial^2}{\partial x^2}\left(H\overline{u}^2 + \frac{g}{2}H^2\right) + g\frac{\partial}{\partial x}\left(H\frac{\partial z_b}{\partial x}\right)\right)\\
&\quad + \frac{H\overline{u}^2}{2}\frac{\partial^2(H+2z_b)}{\partial x^2} - \frac{1}{2}\frac{\partial(H+2z_b)}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{g}{2}H^2\right) + gH\frac{\partial z_b}{\partial x}\right)\\
&= H\left(-\overline{u}\frac{\partial^2(H\overline{u})}{\partial x^2} + \frac{1}{2}\frac{\partial^2(H\overline{u}^2)}{\partial x^2} + \frac{\overline{u}^2}{2}\frac{\partial^2(H+2z_b)}{\partial x^2}\right)\\
&\quad + \frac{gH}{2}\left(H\frac{\partial^2(H+z_b)}{\partial x^2} - 2\frac{\partial z_b}{\partial x}\frac{\partial(H+z_b)}{\partial x}\right).
\end{aligned}$$

Introducing the new variable

$$\overline{q}_{nh} = \sqrt{H}\overline{p}_{nh},$$

relation (62) becomes

$$-4H^2\frac{\partial^2\overline{q}_{nh}}{\partial x^2} + \Lambda\overline{q}_{nh} = 8\sqrt{H}B, \tag{63}$$

that is an non-homogeneous differential equation with

$$\Lambda = 16\left(1 + \left(\frac{\partial z_b}{\partial x}\right)^2\right) - 8H\frac{\partial^2 z_b}{\partial x^2} + 16\frac{\partial H}{\partial x}\frac{\partial z_b}{\partial x} - 2H\frac{\partial^2 H}{\partial x^2} + 3\left(\frac{\partial H}{\partial x}\right)^2.$$

And the sign of Λ in Eq. (63) gives interesting informations about the influence of the non-hydrostatic terms. Indeed, for smooth/small variations of z_b and H, we have $\Lambda > 0$ whereas large variations of z_b and H can lead to the situation where $\Lambda < 0$.

When $\Lambda > 0$, Eq. (63) corresponds to a diffusion type equation and when $\Lambda < 0$, Eq. (63) corresponds to an Helmholtz type equation. This remark is very important since situations where $\Lambda < 0$ may correspond to areas where the non-hydrostatic effects can be significant

4.2. **Requirements for the pressure $\overline{p}$.** The positivity of the pressure p for the incompressible Euler equations (see paragraph 2.5) is an acute problem. On the one hand, the Euler system allows the pressure p to be non-positive, on the other hand $p < 0$ means that the fluid is no more in contact with the bottom and the system (11)-(13),(4),(6) has to be reformulated, especially its boundary conditions.

This problem vanishes when considering the Saint-Venant system. Indeed in this situation, the pressure term corresponds to

$$\frac{g}{2}H^2,$$

that is always non-negative.

When $H \to 0$ the Euler equations, the proposed non-hydrostatic model but also the Saint-Venant system are no more physically relevant. We would like in this situation, as for the Saint-Venant system, that the model (37)-(38) well behaves both at the continuous and discrete level.

5. **Analytical solutions.** The analysis of the proposed non-hydrostatic model being very complex, the knowledge of analytical solutions allows to examine the behavior of the model in particular situations. Moreover, analytical solutions are an important tool for the validation of numerical schemes.

In the following, we propose different analytical solutions for the averaged non-hydrostatic model (37)-(38).

5.1. **Time dependent analytical solutions.** In this paragraph we consider the Euler system (11) with the boundary conditions (4),(6) and (13). This system can also be written under the form

$$
\left\{
\begin{array}{ll}
\dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x} \displaystyle\int_{z_b}^{\eta} u \ dz = 0, & \text{(64a)} \\[4mm]
w = -\dfrac{\partial}{\partial x} \displaystyle\int_{z_b}^{z} u \ dz, & \text{(64b)} \\[4mm]
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} + \dfrac{\partial p}{\partial x} = 0, & \text{(64c)} \\[4mm]
\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} + \dfrac{\partial p}{\partial z} = -g + s, & \text{(64d)}
\end{array}
\right.
$$

coupled with the boundary condition (13) where s is an external forcing term.

And we have the following proposition.

Proposition 3. *Let us consider the variables* u, w, H, z_b, p *defined by*

$$
\left\{
\begin{array}{ll}
H(x,t) = \max\left(H_0 - \dfrac{b_2}{2}\left(x - \displaystyle\int_{\tilde{t}^0}^{t} f(t_1)dt_1 \right)^2, 0 \right), & \text{(65a)} \\[4mm]
u(x,z,t) = f(t)\mathbf{1_{H>0}}, & \text{(65b)} \\[2mm]
w(x,z,t) = b_2 x f(t)\mathbf{1_{H>0}}, & \text{(65c)} \\[2mm]
z_b(x) = b_1 + \dfrac{b_2}{2}x^2, & \text{(65d)} \\[4mm]
p(x,z,t) = (g + b_2 f^2)(H + z_b - z)\mathbf{1_{H>0}}, & \text{(65e)} \\[3mm]
s(x,z,t) = b_2 x \dfrac{df}{dt}, & \text{(65f)}
\end{array}
\right.
$$

where $H_0 > 0, b_1, b_2$ *are constants and the function* f *satisfies the ODE*

$$
\dfrac{df}{dt} + b_2(g + b_2 f^2)\int_{\tilde{t}^0}^{t} f(t_1)dt_1 = 0, \quad f(t_0) = f^0, \ \tilde{t}^0 \in \mathbb{R}. \tag{66}
$$

Then u, w, H, z_b, p *as defined previously satisfy the 2d incompressible Euler equations with free surface* (64) *with the boundary condition* (13) *where* $p^a = 0$.

Proof. The proof relies on simple manipulations. Replacing (65) in (64) shows the solution is analytic when (66) is satisfied. $\square$

Remark 4. Analytical solutions without the source term s in (64d) would have been a stronger result. Nevertheless, since we only consider a source term for one of the four equations (64), it remains an interesting result for numerical validations.

These analytical solutions generalize the solutions obtained by Thacker [41] for the shallow water equations. The analysis of the ODE (66) is not in the scope of

this paper. Notice that the change of variables

$$h(t) = \int_{\tilde{t}^0}^{t} f(t_1)dt_1,$$

allows to rewrite (66) under the form

$$\frac{d^2h}{dt^2} + b_2 \left(g + b_2 \left(\frac{dh}{dt} \right)^2 \right) h = 0,$$

$$h(t_0) = \int_{\tilde{t}^0}^{t^0} f(t_1)dt_1, \quad \dot{y}(t_0) = f(t_0) = f^0.$$

$$(67)$$

It is worth noticing that when $H > 0$ the free surface is a straight line varying with time. Indeed, from the definitions of prop. 3 and when $H > 0$, we get that for any t

$$H + z_b = b_1 - \frac{b_2}{2} \left(-2x \int_{\tilde{t}^0}^{t} f(t_1)dt_1 + \left(\int_{\tilde{t}^0}^{t} f(t_1)dt_1 \right)^2 \right),$$

that is a linear function of the x variable.

The analytical solution depicted in prop. (3) is interesting for two reasons. First, it allows to confront a numerical scheme with behaviors difficult to capture typically drying and flooding. The second reason is explained in the following proposition.

Proposition 4. *The variables H, $\overline{u}$, $\overline{w}$, z_b defined as in Eqs. (65a)-(65d) and $\overline{p}$ defined by*

$$\overline{p} = \frac{g}{2}H + \overline{p}_{nh} = \frac{1}{H} \int_{z_b}^{\eta} p(x,z,t)dz,$$

with p given in (65e) are analytical solutions of the depth-averaged Euler system (37) completed with the source term s.

The propositions 3 and 4 produce a very important consequence. Taking into account the source term s, we have exhibited an analytical solution for the 2d Euler system (11)-(12) with free surface which is also an analytical solution for the non-hydrostatic model (37)-(38) we propose. This a strong argument proving our model is a good approximation of the Euler system for shallow water flows. And this is reinforced by the following proposition.

Proposition 5. *When f satisfies (66), the solution (65) is not an analytical solution of the Green-Naghdi model (50)-(51). If f satisfies the ODE*

$$\frac{df}{dt} + b_2 \left(g + \frac{4b_2}{3}f^2 \right) \int_{\tilde{t}^0}^{t} f(t_1)dt_1 = 0, \quad f(t_0) = f^0, \quad \tilde{t}^0 \in \mathbb{R}, \qquad (68)$$

then (65) is an analytical solution of the Green-Naghdi model (50)-(51). But the energy balance (51) is not consistent with the energy equation (12) of the Euler system.

Proof of prop. 5. The proof relies on simple calculations. Since from (65) we have

$$\int_{z_b}^{\eta} E \, dz = \int_{z_b}^{\eta} \left(\frac{u^2 + w^2}{2} + gz \right) dz = -\frac{b_0 f^2}{4} \left(x - \int_{\tilde{t}^0}^{t} f(t_1)dt_1 \right)^2 \left(1 + b_0^2 x^2 \right)$$

$$+ \frac{g b_0^2}{8} \left(\left(x^2 - \left(x - \int_{\tilde{t}^0}^{t} f(t_1)dt_1 \right)^2 \right)^2 - x^4 \right)$$

$$= \quad \overline{E}$$
$$\neq \quad \overline{E}_{gn},$$

this proves the result. □

To illustrate the difference between the solutions having the form (65) for the two non-hydrostatic models (37) and (50), we plot on Fig. 2 the solutions of (66) and (68). The solutions have been obtained using an implicit first order Euler
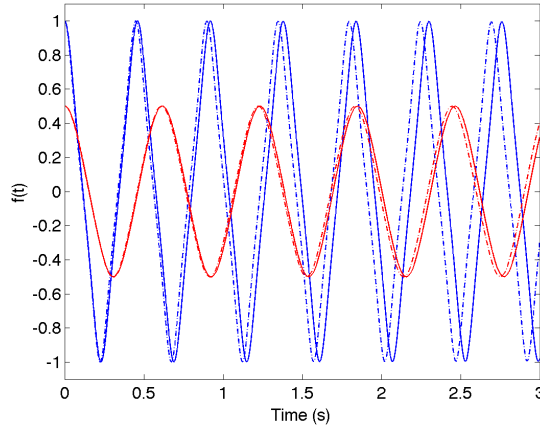

FIGURE 2. Comparison of the solutions of (66) (solid lines) and (68) (dashed lines).

scheme solving (66) and (68). Over Fig. 2, the solid lines (resp. the dashed lines) correspond to solutions of (66) (resp. (68)). The two curves with amplitude 1 have been obtained with $b_0 = 15$, $\tilde{t}_0 = t_0 = 0$ s, $f(t_0) = 1$ and the two curves with amplitude $\frac{1}{2}$ have been obtained with $b_0 = 10$, $\tilde{t}_0 = t_0 = 0$ s, $f(t_0) = \frac{1}{2}$. We observe that whenever the solutions of (66) and (68) remain periodic, the period differs especially for large values of b_0.

5.2. **Solitary wave solutions.** Using a process similar to what is done in [23, 13], in the case where $z_b = cst$, we can exhibit solitary waves for the system (37) under the form

$$
\begin{cases}
H = H_0 + a \left(\text{sech} \left(\frac{x - c_0 t}{l} \right) \right)^2, & \text{(69a)} \\[2ex]
\overline{u} = c_0 \left(1 - \frac{d}{H} \right), & \text{(69b)} \\[2ex]
\overline{w} = -\frac{a c_0 d}{l H} \text{sech} \left(\frac{x - c_0 t}{l} \right) \text{sech}' \left(\frac{x - c_0 t}{l} \right), & \text{(69c)} \\[2ex]
\overline{p}_{nh} = \frac{a c_0^2 d^2}{2 l^2 H^2} \left((2 H_0 - H) \left(\text{sech}' \left(\frac{x - c_0 t}{l} \right) \right)^2 \right. \\[2ex]
\qquad \left. + H \, \text{sech} \left(\frac{x - c_0 t}{l} \right) \text{sech}'' \left(\frac{x - c_0 t}{l} \right) \right), & \text{(69d)}
\end{cases}
$$

where f' denotes the derivative of function f and

$$c_0 = \frac{l}{d}\sqrt{\frac{gH_0^3}{l^2 - H_0^2}}, \quad \text{and} \quad a = \frac{H_0^3}{l^2 - H_0^2},$$

and $(d, l, H_0) \in \mathbb{R}^3$ with $l > H_0 > 0$.

The system (69) also gives analytical solutions for the Green-Naghdi system. Indeed, replacing a and c_0 by a_γ and $c_{0,\gamma}$ defined by

$$a_\gamma = \frac{H_0^3}{\frac{\gamma}{2}l^2 - H_0^2},$$

$$c_{0,\gamma} = \sqrt{\frac{\gamma}{2}}\frac{l}{d}\sqrt{\frac{gH_0^3}{\frac{\gamma}{2}l^2 - H_0^2}},$$

with $(d, l, H_0) \in \mathbb{R}^3$ and $l > H_0 > 0$, the system (69) gives an analytical solution for the general system (53). Therefore, we are able to compare the analytical solutions of the two non-hydrostatic system. On Fig. 3, we have plotted the water depth at three different instants $t_0 = 0$ s, $t_1 = 4$ s and $t_2 = 12$ s corresponding to the propagation of the two analytical solitary waves with $H_0 = 1$ m and $d = 2$ m. Fig. 3-*(a)*, the analytical solutions of the depth-averaged model and the Green-Naghdi model are depicted by the solid and dashed lines, respectively. The solutions correspond to the choice $l = 2$ m and the corresponding values of a_2 and $c_{0,2}$ for the depth-averaged model and the corresponding values of $a_{3/2}$ and $c_{0,3/2}$ for the Green-Naghdi model. We see on Fig. 3-*(a)* that starting from the same physical parameters H_0, d, l, the two non-hydrostatic models propagate two solitons but with different amplitudes and propagation velocities. On the contrary, we can choose the physical parameters, typically l, so that the two solitons have the same amplitude and propagation velocities. Indeed, choosing for the depth-averaged system $l = 2$ m and the corresponding values of a_2 and $c_{0,2}$ and for the Green-Naghdi model $l = \frac{4}{\sqrt{3}}$ m and the corresponding values of $a_{3/2}$ and $c_{0,3/2}$ we obtained on Fig. 3-*(b)* two solitons with the same amplitude and propagation velocities but a slightly different shape.
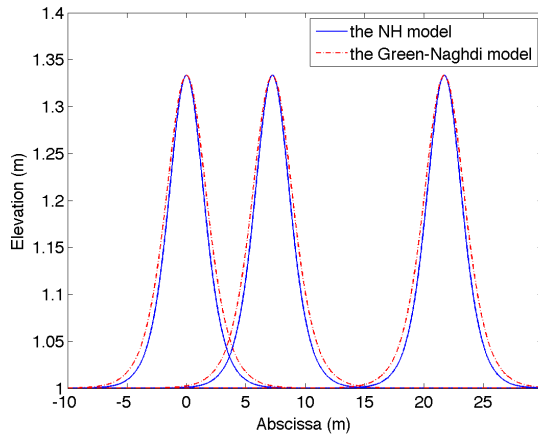
5.3. Stationary solutions.

5.3.1. *Regularity of stationary solutions.* Simple manipulations show that stationary analytical solutions of (37) have to satisfy

$$\begin{cases} H\overline{u} = Q_0 = Cst, & \text{(70a)} \\[2mm] \dfrac{\partial}{\partial x}\left(\dfrac{Q_0^2}{H} + \dfrac{g}{2}H^2 + H\overline{p}_{nh}\right) = -(gH + 2\overline{p}_{nh})\dfrac{\partial z_b}{\partial x}, & \text{(70b)} \\[2mm] H\overline{w} = \dfrac{Q_0}{2}\dfrac{\partial}{\partial x}(H + 2z_b), & \text{(70c)} \\[2mm] \overline{p}_{nh} = \dfrac{Q_0}{2}\dfrac{\partial \overline{w}}{\partial x}, & \text{(70d)} \end{cases}$$

or equivalently

(a)

(b)
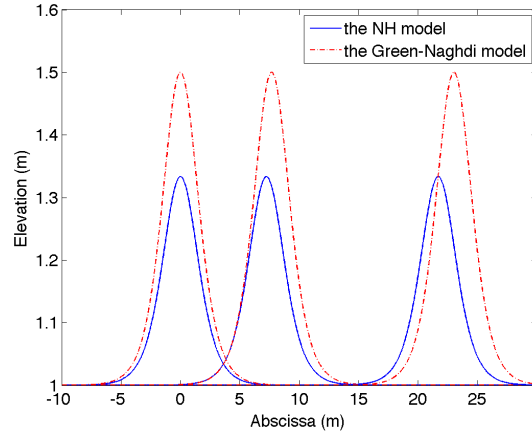
FIGURE 3. Comparaison of analytical solutions of the depth-averaged model (NH) (solid lines) and of the Green-Naghdi (dashed lines) in the case of a solitary wave: *(a)* same values of l *(b)* same amplitude and propagation velocity.

$$\begin{cases} \dfrac{\partial H}{\partial x} = \dfrac{2}{Q_0} H \overline{w} - 2 \dfrac{\partial z_b}{\partial x} \\[2mm] \dfrac{\partial \overline{w}}{\partial x} = \dfrac{2}{Q_0} \overline{p}_{nh}, \\[2mm] \dfrac{\partial \overline{p}_{nh}}{\partial x} = \left(\dfrac{Q_0^2}{H^2} - gH - \overline{p}_{nh} \right) \left(\dfrac{2}{Q_0} \overline{w} - \dfrac{2}{H} \dfrac{\partial z_b}{\partial x} \right) - \left(g + \dfrac{2\overline{p}_{nh}}{H} \right) \dfrac{\partial z_b}{\partial x}, \end{cases}$$

and $\overline{u} = \frac{Q_0}{H}$. Hence, as long as $H > 0$, we have $(H, \overline{w}, \overline{p}_{nh}) \in (C^k)^3$ if $z_b \in C^k$. This means that when z_b is at least continuous, the stationary solutions of the non-hydrostatic model are necessarily continuous and do not admit shocks.

5.3.2. *Stationary quasi-analytical solutions.* From the previous writing, we deduce the following proposition.

Proposition 6. *Choosing Q_0, a boundary condition H_0 for H and a given function $f = f(x)$ corresponding to the desired vertical velocity i.e. $\overline{w} = f$, then the variables $\overline{p}_{nh}, H, z_b, \overline{u}$, defined by*

$$\begin{cases} \overline{p}_{nh} = \dfrac{Q_0}{2} \dfrac{\partial f}{\partial x}, & \text{(72a)} \\[2mm] \left(\dfrac{g}{2} H - \dfrac{Q_0^2}{H^2} \right) \dfrac{\partial H}{\partial x} = -\dfrac{H}{Q_0} \left(gH + Q_0 \dfrac{\partial f}{\partial x} \right) f - \dfrac{Q_0}{2} H \dfrac{\partial^2 f}{\partial x^2}, & \text{(72b)} \\[2mm] \dfrac{\partial z_b}{\partial x} = -\dfrac{1}{2} \dfrac{\partial H}{\partial x} + \dfrac{Hf}{Q_0}, & \text{(72c)} \\[2mm] \overline{u} = \dfrac{Q_0}{H}, & \text{(72d)} \end{cases}$$

are stationary quasi-analytical of the system (37).

The word "quasi-analytical" refers to the fact that the previous set of equations only contains two simple ODEs that have to be solved numerically.

Proof of proposition 6. The proof is very simple, it only consists in a reformulation of the system (70a)-(70c) with the assumption $\overline{w} = f$, f given. □

Remark 5. Since the quantity

$$\frac{g}{2} H - \frac{Q_0^2}{H^2},$$

appears in the ODE to solve (72d), it is possible to obtain solutions for H with discontinuities. But necessarily, due to the second equation to solve, discontinuities also appears over z_b. Thus, this is not contradictory with the results in paragraph 5.3.

As in paragraphs 5.1 and 5.2, we compare the stationary solutions for the depth-averaged model (37) and the Green-Naghdi system (50). Following prop. 6, we can also exhibit stationary quasi-analytical solutions for the Green-Naghdi system (50). For any given enough smooth function, f and the solutions of the system

$$\begin{cases} \overline{p}_{nh} = \dfrac{Q_0}{2} \dfrac{\partial f}{\partial x}, & \text{(73a)} \\[2mm] \left(\dfrac{g}{2} H - \dfrac{Q_0^2}{H^2} \right) \dfrac{\partial H}{\partial x} = -\dfrac{H}{Q_0} \left(gH + \dfrac{4Q_0}{3} \dfrac{\partial f}{\partial x} \right) f - \dfrac{2}{3} Q_0 H \dfrac{\partial^2 f}{\partial x^2}, & \text{(73b)} \\[2mm] \dfrac{\partial z_b}{\partial x} = -\dfrac{1}{2} \dfrac{\partial H}{\partial x} + \dfrac{Hf}{Q_0}, & \text{(73c)} \\[2mm] \overline{u} = \dfrac{Q_0}{H}, & \text{(73d)} \end{cases}$$

are analytical solutions of the Green-Naghdi system (50). Numerical comparisons between the solutions of systems (72) and (73) are given in the following paragraph.

5.3.3. *Numerical illustrations.* To illustrate the analytical solutions described by prop. 6, we give below two typical examples. The analytical solutions are obtained choosing

$$f(x) = 2c(x - a)e^{-b(x-a)^2}, \tag{74}$$

and correspond to a channel of length $L = 10$ m where we impose the inflow $Q_0 > 0$ at the entrance (left boundary) and the water depth H_0 at the exit (right boundary).

For Fig. 4, the following parameters values $Q_0 = 1.8 \ m^2.s^{-1}$, $H_0 = 1 \ m$, $a = 5 \ m$, $b = 3.4 \ m^{-2}$ and $c = 1.5 \ s^{-1}$ are considered. On Fig. 4-(a), we compare the free surface $\eta = H + z_b$ obtained with the quasi-analytical solution (72b),(72d) of the non-hydrostatic model to the one obtained with the Saint-Venant system (with the same topography z_b and the same boundary conditions). Likewise on Fig. 4-(b), we compare the velocity field $\overline{u}$ obtained with the depth-averaged Euler model to the one obtained with the Saint-Venant system (with the same topography z_b and the same boundary conditions). The velocity field $\overline{w}$ corresponding to the depth-averaged system is also plotted on Fig. 4-(b). Over Fig. 4-(c), we compare the total pressure $gH/2 + \overline{p}_{nh}$ to its hydrostatic part $gH/2$.

Figure 5 is similar to Figure 4 but has been obtained with the parameters values $Q_0 = 1.35 \ m^2.s^{-1}$, $a = 5 \ m$, $b = 4.6 \ m^{-2}$ and $c = 1.0 \ m^{-1}$. Figures 4 and 5 emphasize the influence of the non-hydrostatic effects.

On Fig. 6, we compare the quasi-analytical solutions of the systems (72) and (73). The definition of the function f is still given by (74). For the inflow Q_0, we have chosen $Q_0 = 1.3 \ m^2.s^{-1}$, the boundary condition H_0 and the parameters a, b and c have the same values as for Fig. 5. Notice that the resolution of (73) with exactly the same parameters values as those used for Fig. 5 i.e. with $Q_0 = 1.35 \ m^2.s^{-1}$ instead of $Q_0 = 1.3 \ m^2.s^{-1}$ leads to a discontinuous solution (see remark 5). In the Green-Naghdi model, the amplitude of the waves is higher than in the depth-averaged model.

6. **Conclusion.** In this paper we have proposed a shallow water type model integrating the non-hydrostatic effects. The derivation process is based on a minimization principle and suitable closure relations.

The proposed depth-averaged Euler system has interesting properties

- the model formulation only involves first order partial derivatives,
- the derivation process naturally provides with an expression for the topography source terms,
- the proposed model is similar to the well-known Green-Naghdi model but gives a natural expression of the topography source term,
- starting from the Navier-Stokes system instead of the Euler system, a depth-averaged version of the Navier-Stokes system is obtained integrating the viscous/friction effects.

Since the pressure terms are not necessarily non negative, the behavior of the averaged model when the water depth tends to zero has to be clarified. The derivation of an efficient and robust numerical scheme able to treat theses situations is under study.
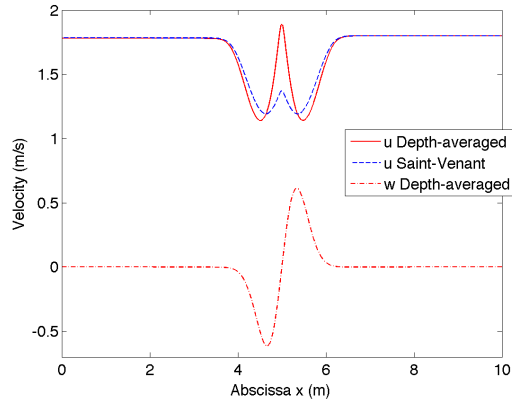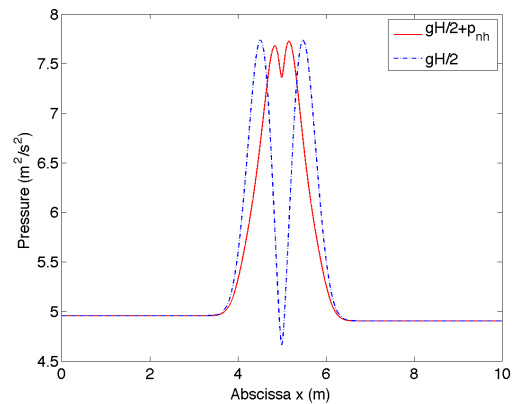
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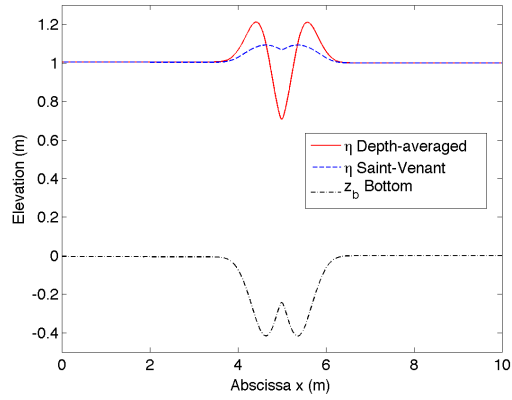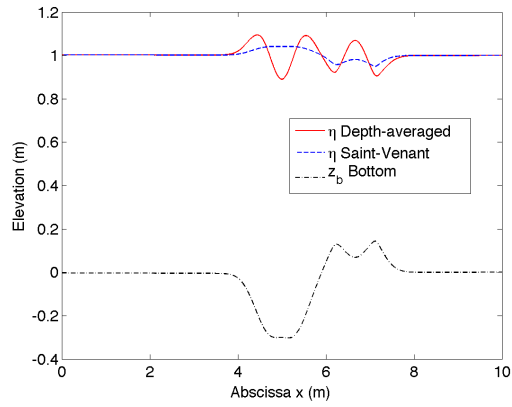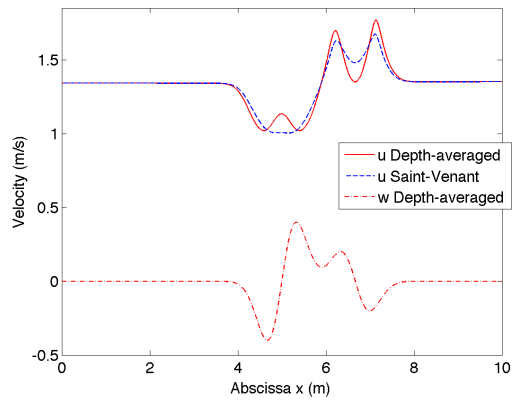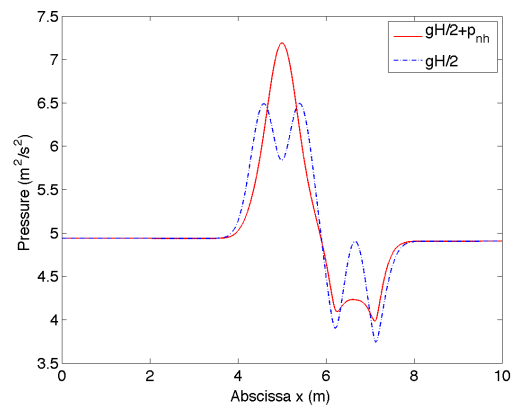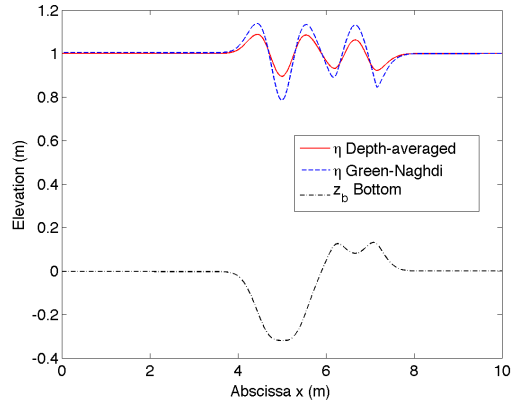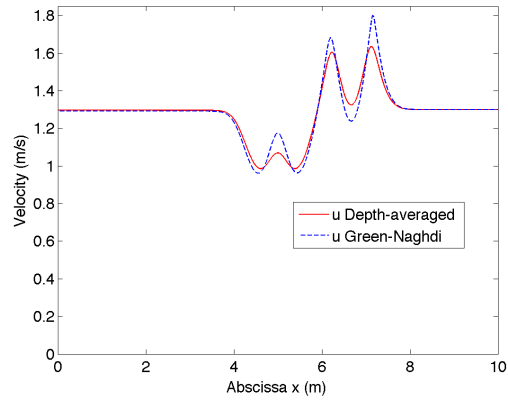
FIGURE 4. Analytical solutions - Comparison of Saint-Venant and depth-averaged Euler solutions: *(a)* free surface $H + z_b$ and bottom profile z_b, *(b)* velocities $\overline{u}$ and $\overline{w}$ and *(c)* total pressure $gH/2 + \overline{p}_{nh}$ and hydrostatic part of the pressure $gH/2$.
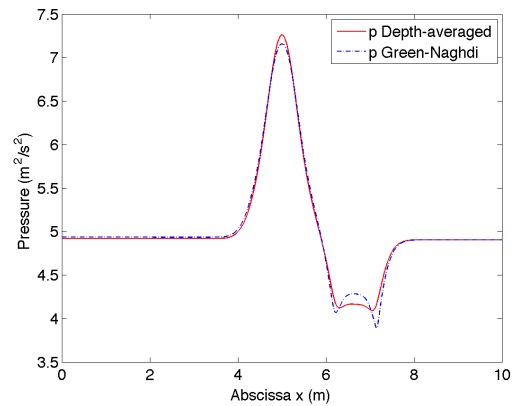
FIGURE 5. Analytical solutions- Comparison of Saint-Venant and depth-averaged Euler solutions: *(a)* free surface $H + z_b$ and bottom profile z_b, *(b)* velocities $\overline{u}$ and $\overline{w}$ and *(c)* total pressure $gH/2 + \overline{p}_{nh}$ and hydrostatic part of the pressure $gH/2$.

FIGURE 6. Analytical solutions- Comparison of Green-Naghdi and depth-averaged Euler solutions: (a) free surface $H + z_b$ and bottom profile z_b, (b) velocities $\overline{u}$ and (c) total pressures $gH/2 + \overline{p}_{nh}$.

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