Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 336 (2003) 531–536
COMPTES RENDUS MATHEMATIQUE

Mathematical Problems in Mechanics/Partial Differential Equations

A new model of Saint Venant and Savage–Hutter type for gravity driven shallow water flows

Un nouveau modèle de type Saint Venant et Savage–Hutter pour les écoulements gravitaires en eaux peu profondes

François Bouchut [a], Anne Mangeney-Castelnau [c], Benoît Perthame [a,b], Jean-Pierre Vilotte [c]

[a] *Département de mathématiques et applications, CNRS & École normale supérieure, 45, rue d'Ulm, 75230 Paris cedex 05, France*
[b] *INRIA Rocquencourt, projet M3N, domaine de Voluceau, BP 105, 78153 Le Chesnay cedex, France*
[c] *Département de modélisation physique et numérique, IPGP, 4, pl. Jussieu, 75232 Paris cedex 05, France*

Received 15 November 2002; accepted after revision 18 February 2003

Presented by Pierre-Louis Lions

Abstract

We introduce a new model for shallow water flows with non-flat bottom. A prototype is the Saint Venant equation for rivers and coastal areas, which is valid for small slopes. An improved model, due to Savage–Hutter, is valid for small slope variations. We introduce a new model which relaxes all restrictions on the topography. Moreover it satisfies the properties (i) to provide an energy dissipation inequality, (ii) to be an exact hydrostatic solution of Euler equations. The difficulty we overcome here is the normal dependence of the velocity field, that we are able to establish exactly. Applications we have in mind concern, in particular, computational aspects of flows of granular material (for example in debris avalanches) where such models are especially relevant. ***To cite this article: F. Bouchut et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous introduisons un nouveau modèle pour les écoulements en eaux peu profondes avec fond variable. Un prototype est l'équation de Saint Venant pour les rivières et zones côtières, qui est valable pour des pentes faibles. Un modèle amélioré, dû à Savage–Hutter, est valable pour de petites variations de pente. Nous introduisons un nouveau modèle valable quelle que soit la topographie, et qui a les propriétés (i) de fournir une inégalité de dissipation d'énergie, (ii) d'être une solution hydrostatique exacte des équations d'Euler. La difficulté que nous résolvons est la dépendance du champ de vitesse dans la variable normale au fond, que nous sommes capables d'établir exactement. Les applications visées sont le calcul numérique des écoulements granulaires (par exemple avalanches de débris) pour lequel un tel modèle est particulièrement adapté. ***Pour citer cet article : F. Bouchut et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

E-mail addresses: fbouchut@dma.ens.fr (F. Bouchut), mangeney@ipgp.jussieu.fr (A. Mangeney-Castelnau), perthame@dma.ens.fr (B. Perthame), vilotte@ipgp.jussieu.fr (J.-P. Vilotte).

1631-073X/03/$ – see front matter
doi:10.1016/S1631-073X(03)00117-1

Version française abrégée

Les écoulements gravitaires en eaux peu profondes, survenant dans l'océan, les rivières, ou les avalanches de débris, peuvent être décrits à plusieurs niveaux d'approximation. En plus de l'hypothèse de faible profondeur, le modèle de Saint Venant (1) suppose un fond en pente faible, tandis que le modèle de Savage–Hutter (3) suppose une faible variation de pente.

Le modèle que nous introduisons ci-dessous est plus précis, et ne fait pas d'hypothèse sur la topographie. Il a également la propriété de vérifier une inégalité de dissipation d'énergie, et de préserver les équilibres du lac au repos. Un développement limité dans la variation de pente permet de retrouver le modèle de Savage–Hutter (3), mais avec un terme supplémentaire en $\theta_X H^2$ permettant de restaurer la dissipation d'énergie et la conservation des équilibres.

Considérons pour un modèle unidimensionnel une variable d'abscisse curviligne X associée à la géométrie du fond, et l'angle $\theta(X)$ entre l'horizontale et la tangente au fond, comme indiqué sur la Fig. 1. Ces variables sont reliées à la coordonnée horizontale x et à la topographie $b(x)$ par les relations (2). Notons $H(t,X)$ l'épaisseur de la couche fluide dans la direction normale au fond, et $u(t,X)$ la vitesse tangentielle, évaluée au contact du fond. Le modèle s'écrit

$$\begin{cases} \frac{\partial}{\partial t}\big(H-\theta_X\frac{H^2}{2}\big)+\frac{\partial}{\partial X}\big(\frac{\ln(1-H\theta_X)}{-\theta_X}u\big)=0,\\ \frac{\partial}{\partial t}u+\frac{\partial}{\partial X}\big(\frac{u^2}{2}\frac{1}{(1-H\theta_X)^2}+Hg\cos\theta+gb\big)=0, \end{cases} \tag{0.1}$$

où g est la constante de la gravitation.

Théorème 0.1. *Le système* (0.1) *a les propriétés suivantes*

(i) *il admet une inégalité de dissipation d'énergie,*

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\frac{\ln(1-H\theta_X)}{-\theta_X}\frac{u^2}{2}+\left(\frac{H^2}{2}-\theta_X\frac{H^3}{3}\right)g\cos\theta+\left(H-\theta_X\frac{H^2}{2}\right)gb\right]\\ &\quad+\frac{\partial}{\partial X}\left[\frac{\ln(1-H\theta_X)}{-\theta_X}\left(\frac{u^2}{2}\frac{1}{(1-H\theta_X)^2}+Hg\cos\theta+gb\right)u\right]\leqslant 0, \end{aligned} \tag{0.2}$$

(ii) *il préserve l'état stationnaire du lac au repos $u=0$, $H\cos\theta+b=Cste$,*

(iii) *il donne une solution exacte au système d'Euler à surfacelibre avec hypothèse hydrostatique* (*voir* (12), (13) *ci-dessous*).

Le point clé du théorème ci-dessus est l'obtention de la solution du point (iii) en posant une dépendance explicite dans la variable normale au fond Z dans les formules (15). Contrairement au modèle de Savage–Hutter, la vitesse tangentielle $U(t,X,Z)$ dépend vraiment de Z, par l'effet de la courbure du fond θ_X.

1. Introduction

In this Note we consider gravity driven flows in the shallow water regime. These kinds of regimes arise in many physical situations such as ocean modeling (Ghil [5]), flows in rivers or coastal areas, debris avalanches, etc. For flat bottom, the situation is simpler and the mathematical theory begins to be well settled (see Lions, Perthame and Souganidis [10] for existence of solutions to the Saint Venant system in 1D and hyperbolic case and the extension by Gwiazda [7] for granular materials, Lions [9] for weak solutions to Navier–Stokes system with free boundary, Bresch and Desjardins [1] for viscous Saint Venant system in multidimensions). But for more complex physical situations, the rigorous derivations of systems is still a topic of current interest. For instance the derivation of the conservative form of Saint Venant system with viscosity terms can be found in Gerbeau–Perthame [4]. Models for landslides have been established in Hild et al. [8] and further references can be found in [11].

Here, we address the question of varying topography as it appears in many applications and especially debris avalanches. Several models of shallow water type have been proposed and the most classical are those of Saint Venant and Savage–Hutter. They both assume limitations on the topography. Our purpose is to explain precisely these limitations, and thus to compare the models, but also to introduce a new model where restrictions on the topography are completely dropped and only shallow water limitations remain. An expansion in the slope variation of our model allows to recover the Savage–Hutter model, while an expansion in the slope itself gives the Saint Venant equations. Our analysis relies on the explicit computation of the dependence of the fluid velocity in the normal direction when solving exactly the hydrostatic Euler system. We take into account the curvature of the bottom, which is usually neglected giving a roughly constant velocity in the depth of water.

The classical Saint Venant system [12] for shallow water has been widely validated. It assumes a very slowly varying topography $b(x)$ (x denotes a coordinate in the horizontal direction) and describes the height of water $h(t,x)$, and the water velocity $u(t,x)$ in the direction parallel to the bottom. It uses the following equations (we restrict to one space dimension for simplicity but two dimensions will be addressed later)

$$\text{(S–V)}\qquad \begin{cases} \frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hu) = 0, \\ \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\big(hu^2 + \frac{g}{2}h^2\big) = -hgb_x. \end{cases} \tag{1}$$

Here g denotes the gravity constant. This model is very robust, being hyperbolic and admitting a convex entropy (the energy). Another good property is that it preserves the steady state of a lake at rest $h+b=Cst$, $u=0$. However, as we shall see below, its validity is restricted to small slopes and thus it can be considered as a first-order system in the 'small parameter' b_x (thus it satisfies the hydrostatic Euler equations only for flat bottom $b_x=0$).

In order to state more accurate models and to take into account the variations of the bottom slope, we introduce, as illustrated on Fig. 1, the angle $\theta(X)$ of the bottom tangent with the horizontal reference and the curvilinear coordinate X, so that

$$\cos\theta\,\mathrm{d}X = \mathrm{d}x, \qquad b_x = \tan\theta, \qquad b_X = \sin\theta. \tag{2}$$

Let us first consider the following variant of Savage–Hutter system [13,6],

$$\text{(H–S)}\qquad \begin{cases} \frac{\partial}{\partial t}H + \frac{\partial}{\partial X}(Hu) = 0, \\ \frac{\partial}{\partial t}(Hu) + \frac{\partial}{\partial X}\big(Hu^2 + \frac{H^2}{2}g\cos\theta\big) = -g\sin\theta\big(H - \theta_X\frac{H^2}{2}\big). \end{cases} \tag{3}$$

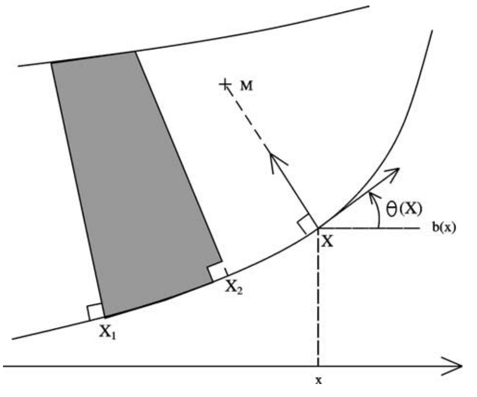

Fig. 1. The fluid layer.

Fig. 1. La couche de fluide.

Now $H(t,X)$ denotes the width of fluid in the normal direction at a base point X, and $u(t,X)$ denotes the tangential velocity (the normal velocity being very small due to the shallow water and the non penetration conditions). This system is still hyperbolic and it has two properties that are inherited from the Navier–Stokes system.

Proposition 1.1. *The system* (3) *has the properties*

(i) *it admits an entropy dissipation inequality, which is just the energy conservation for smooth solutions,*

$$\frac{\partial}{\partial t}\left(H\frac{u^2}{2}+\frac{H^2}{2}g\cos\theta+Hgb\right)+\frac{\partial}{\partial X}\left[\left(H\frac{u^2}{2}+H^2g\cos\theta+Hgb\right)u\right]\leqslant 0, \tag{4}$$

(ii) *it preserves the steady state of a lake at rest* $u=0$, $H\cos\theta+b=Cst$,

(iii) *for constant slope* $\theta_X=0$, *the system* (3) *gives an exact solution to the free surface Euler system with hydrostatic assumption.*

Proof. Combining the equations of (3) and using that $\sin\theta=b_X$, we get another equation on u,

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial X}\left(\frac{u^2}{2}+Hg\cos\theta+gb\right)=0. \tag{5}$$

Multiplying the first equation in (3) by $u^2/2+Hg\cos\theta+gb$, multiplying (5) by Hu and adding the results gives (i). The property (ii) is obvious in the first equation of (3) and in (5). Finally, we check (iii). For a constant slope we denote by Z a coordinate orthogonal to the bottom. The hydrostatic Euler system holds in the domain $0<Z<H(t,X)$,

$$\begin{cases}\frac{\partial}{\partial X}u+\frac{\partial}{\partial Z}w=0, \quad w(t,X,Z=0)=0,\\ \frac{\partial}{\partial t}u+u\frac{\partial}{\partial X}u+w\frac{\partial}{\partial Z}u+\frac{\partial}{\partial X}p=-g\sin\theta,\\ \frac{\partial}{\partial Z}p=-g\cos\theta, \quad p\big(t,X,Z=H(t,X)\big)=0,\end{cases} \tag{6}$$

with the free surface $Z=H$ advected by the flow. We recall that this system is itself derived from the full Euler equation with the approximation $\frac{\partial}{\partial Z}w=\mathrm{O}(1)$ and $w=\mathrm{O}(\varepsilon)$, where we have used aspect ratio $H/L\simeq\varepsilon$, where L is the typical length of the phenomena in X.

Indeed, for compatible initial data, a solution is simply $u=u(t,X)$, $w=-Z\frac{\partial u}{\partial X}$, $p=(H(t,X)-Z)g\cos\theta$, and this yields (5). The equation on H in (3) is also exact because u is constant in the normal direction. □

Our model (3) differs from the original Savage–Hutter model only through the term in $\theta_X\frac{H^2}{2}$ in the right-hand side. We can remark that θ_X is the curvature of the bottom, and is positive for convex topography. Although this term can be thought as negligible in terms of magnitude, we put it to obtain the energy inequality and to preserve the lake at rest, a property that is useful for numerical tests [2,11]. We can compare also our model (3) or the original Savage–Hutter model with Saint Venant system. They differ even in the case $\theta=Cst$ because there is no simple relation between the height of fluid h (with respect to the vertical) and the width of the fluid layer H (in the normal direction). This explains why the Saint Venant system (1) is valid only for b_X (or θ) of order ε, and it can be obtained by neglecting terms in ε^3 in (3).

2. Velocity dependence in the normal direction

We can give a more accurate model which does not use any asymptotic analysis on the topography but only the shallow water assumption. It involves a nontrivial dependence of the tangential velocity U in the normal variable Z, see (15). A nonconstant dependence was also considered in [3]. Our model also has the advantage of addressing the following shortcoming of model (3). By opposition to the Saint Venant model, the equation on H in (3) does not express the mass conservation exactly. Because of curvature effects the volume of fluid between two points X_1 and

X_2 is not $\int_{X_1}^{X_2} H(t,X)\,\mathrm{d}X$, but rather $\int_{X_1}^{X_2}(H-\theta_X\frac{H^2}{2})\,\mathrm{d}X$ (see Fig. 1). The nonnegativity of the volume element $H-\theta_X\frac{H^2}{2}$ expresses an expected smallness condition on the width of fluid for the validity of the parametrization by X. Our generalized model takes this into account, and reads

$$\begin{cases} \frac{\partial}{\partial t}\big(H-\theta_X\frac{H^2}{2}\big)+\frac{\partial}{\partial X}\big(\frac{\ln(1-H\theta_X)}{-\theta_X}u\big)=0,\\ \frac{\partial}{\partial t}u+\frac{\partial}{\partial X}\big(\frac{u^2}{2}\frac{1}{(1-H\theta_X)^2}+Hg\cos\theta+gb\big)=0. \end{cases} \tag{7}$$

Notice that for the sake of simplicity we have used the velocity equation. The equation on the momentum Hu can be written also. However, a difficulty occurs here in the fact that this equation is not in conservative form if $\theta_X\neq 0$.

Theorem 2.1. *The system* (7) *has the properties*

(i) *it admits an entropy dissipation inequality,*

$$\frac{\partial}{\partial t}\left[\frac{\ln(1-H\theta_X)}{-\theta_X}\frac{u^2}{2}+\left(\frac{H^2}{2}-\theta_X\frac{H^3}{3}\right)g\cos\theta+\left(H-\theta_X\frac{H^2}{2}\right)gb\right] + \frac{\partial}{\partial X}\left[\frac{\ln(1-H\theta_X)}{-\theta_X}\left(\frac{u^2}{2}\frac{1}{(1-H\theta_X)^2}+Hg\cos\theta+gb\right)u\right]\leqslant 0, \tag{8}$$

(ii) *it preserves the steady state of a lake at rest $u=0$, $H\cos\theta+b=Cst$,*

(iii) *it gives an exact solution to the free surface Euler system with hydrostatic assumption* (*see* (12), (13) *below*).

Proof. We use the coordinates (X,Z), where Z is a variable normal to the bottom, $0<Z<H(t,X)$ (see Fig. 1). We denote by M a generic point in the fluid, represented in cartesian coordinates, $M=(x,b(x))+Z(-\sin\theta,\cos\theta)$. We have

$$\nabla_{X,Z}M=\begin{pmatrix} J\cos\theta & -\sin\theta\\ J\sin\theta & \cos\theta\end{pmatrix},\qquad J=1-Z\theta_X=\det\nabla_{X,Z}M. \tag{9}$$

The positivity of J expresses an admissibility condition on the system of coordinates. The Euler equations written on the tangential velocity U and normal velocity W in the local frame defined by the bottom are (exactly)

$$\begin{cases} \frac{\partial}{\partial X}U+\frac{\partial}{\partial Z}(JW)=0,\\ \frac{\partial}{\partial t}(JU)+U\frac{\partial}{\partial X}U+JW\frac{\partial}{\partial Z}U+\frac{\partial}{\partial X}P=-Jg\sin\theta+UW\theta_X,\\ \frac{\partial}{\partial t}(JW)+U\frac{\partial}{\partial X}W+JW\frac{\partial}{\partial Z}W+J\frac{\partial}{\partial Z}P=-Jg\cos\theta-U^2\theta_X. \end{cases} \tag{10}$$

It is completed by the conditions $W(Z=0)=0$, and $P(Z=H)=0$. The fluid region is defined by the function

$$\chi(t,X,Z)=\mathbb{I}_{\{0<Z<H(t,X)\}}, \tag{11}$$

which is advected by the flow

$$\frac{\partial}{\partial t}(\chi J)+\frac{\partial}{\partial X}(\chi U)+\frac{\partial}{\partial Z}(\chi JW)=0. \tag{12}$$

The *hydrostatic approximation* still consists in neglecting the normal acceleration, which is justified by assuming that $\frac{\partial}{\partial Z}W=\mathrm{O}(1)$, which implies that W is of order ε (aspect ratio). We therefore replace (10) by

$$\begin{cases} \frac{\partial}{\partial X}U+\frac{\partial}{\partial Z}(JW)=0,\quad W(t,X,Z=0)=0,\\ \frac{\partial}{\partial t}(JU)+U\frac{\partial}{\partial X}U+JW\frac{\partial}{\partial Z}U+\frac{\partial}{\partial X}P=-Jg\sin\theta+UW\theta_X,\\ J\frac{\partial}{\partial Z}P=-Jg\cos\theta-U^2\theta_X,\quad P(t,X,Z=H)=0. \end{cases} \tag{13}$$

This system admits an energy equality (dissipation terms are not considered here)

$$\frac{\partial}{\partial t}\left[\left(\frac{U^2}{2}-\vec{g}\cdot\vec{M}\right)J\right]+\frac{\partial}{\partial X}\left[\left(\frac{U^2}{2}+P-\vec{g}\cdot\vec{M}\right)U\right]+\frac{\partial}{\partial Z}\left[\left(\frac{U^2}{2}+P-\vec{g}\cdot\vec{M}\right)JW\right]=0, \tag{14}$$

where the scalar product can be computed as $\vec{g}\cdot\vec{M}=-g(b+Z\cos\theta)$.

Theorem 2.1 is another statement of the following remark. The solution of this hydrostatic system (with compatible initial data) is given by

$$U(t,X,Z)=\frac{u(t,X)}{1-Z\theta_X},\qquad P=(H-Z)g\cos\theta+\frac{u^2}{2}\left(\frac{1}{(1-H\theta_X)^2}-\frac{1}{(1-Z\theta_X)^2}\right), \tag{15}$$

with u solution to (7). This can be seen as follows. Setting $JU=u$ independent of Z (recall that $J=1-Z\theta_X$), the last equation in (13) can be written $\partial_Z P=-g\cos\theta-u^2\theta_X/J^3$. An integration in Z with the boundary condition in (13) gives the value of P given in (15). Then, since $J\partial_Z U=U\theta_X$ and $\sin\theta=b_X$, we can write the second equation in (13) as

$$\frac{\partial}{\partial t}u+\frac{\partial}{\partial X}\left(\frac{U^2}{2}+P+Zg\cos\theta+gb\right)=0, \tag{16}$$

which gives the second equation in (7). The first equation in (7) is obtained by integration in Z of (12), and this gives point (iii). Then, (i) is deduced from (14) by integration in Z. It can be obtained also directly from (7) by multiplying the two equations by $\frac{u^2}{2}\frac{1}{(1-H\theta_X)^2}+Hg\cos\theta+gb$ and $\frac{\ln(1-H\theta_X)}{-\theta_X}u$ respectively. Finally, (ii) is obvious. □

We can observe that if we make the assumption of small slope variation $\theta_X=\mathrm{O}(\varepsilon)$ and neglect terms in ε^3 and ε^2 respectively in (7), we recover the Savage–Hutter model (3), (5).

Acknowledgements

The authors thank Marie-Odile Bristeau for fruitful discussions during the preparation of this work. This work has been partially supported by the CNRS ACI “Modélisation de processus hydrauliques à surface libre en présence de singularités”.

References

[1] D. Bresch, B. Desjardins, On a viscous shallow water equations (Saint Venant model) and its quasi-geostrophic limit, Preprint.

[2] F. Bouchut, An introduction to finite volume methods for hyperbolic systems of conservation laws with source, École CEA–EDF–INRIA Écoulements peu profonds à surface libre, 7–10 octobre 2002, INRIA Rocquencourt, available at http://www.dma.ens.fr/~fbouchut/publications/fvcours.ps.gz.

[3] S. Douady, B. Andreotti, A. Daerr, On granular surface flow equations, Eur. Phys. J. B 11 (1999) 131–142.

[4] J.-F. Gerbeau, B. Perthame, Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation, DCDS(B) 1 (1) (2001).

[5] M. Ghil, R. Vautard, Nonlinear Climate Theory, Cambridge University Press, Cambridge, London, 2000.

[6] J.M.N.T. Gray, M. Wieland, K. Hutter, Gravity driven free surface flow of granular avalanches over complex basal topography, Proc. Roy Soc. London Ser. A 455 (1999) 1841–1874.

[7] P. Gwiazda, An existence result for a model of granular material with non-constant density, Asymptotic Anal. 30 (2002) 43–60.

[8] P. Hild, I.R. Ionescu, T. Lachand-Robert, I. Rosca, The blocking of an inhomogeneous Bingham fluid. Applications to landslides, Math. Modelling Numer. Anal., to appear.

[9] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Incompressible Models, in: Oxford Lecture Series in Math. Appl., Vol. 1, Oxford University Press, 1996.

[10] P.-L. Lions, B. Perthame, P.E. Souganidis, Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, Comm. Pure Appl. Math. 49 (1996) 599–638.

[11] A. Mangeney, J.-P. Vilotte, M.-O. Bristeau, B. Perthame, C. Simeoni, S. Yerneni, Numerical modeling of avalanches based on Saint Venant equations using a kinetic scheme, Preprint, 2002.

[12] A.J.C. de Saint-Venant, Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit, C. R. Acad. Sci. Paris 73 (1871) 147–154.

[13] S.B. Savage, K. Hutter, The dynamics of avalanches of granular materials from initiation to run-out, Acta Mech. 86 (1991) 201–223.