Mathematical Geology, Vol. 33, No. 2, 2001

Short Note

Precision and Convergence of a Steady Two-Dimensional Ice Sheet Flow Model[1]

Anne Mangeney[2]

Ice-sheet flow is described by the Navier–Stokes equations. We present here an analytical solution for a very simple configuration of two-dimensional ice sheet flow. It is obtained for an imposed flat surface elevation and for a linear flow law. This analytical solution is used here to estimate the performance and precision of a two-dimensional ice sheet flow model. In particular, the comparison of this 2D ice sheet flow model with the analytical solution has allowed to test all the terms involved in the mechanical equations. This analytical solution may be very useful to test similar types of models.

KEY WORDS: hydrodynamics, viscosity, two-dimensional structures, finite-difference methods, analytic.

INTRODUCTION

The flow of ice, lava, or the folding of lithospheric sheets are described by the incompressible Navier–Stokes equations. Approximations are generally made to obtain analytical or simplified solutions such as the long wave approximation (i.e., shallow ice, shallow water approximation) with which the results of a model solving the complete set of mechanical equations can be compared (see, e.g., Hutter, 1983; Mangeney and Califano, 1998). Actually, the shallow ice approximation provides solution for varying free surface and bedrock topography, but does not give an exact evaluation of the field. We present here an analytical solution for a very simple configuration of steady-state two-dimensional ice sheet flow, with particular boundary conditions. It is obtained for an imposed flat surface elevation and for a linear flow law. The equations are solved by introducing a velocity

[1]Received 14 August 1999; accepted 14 March 2000.
[2]Departement de Sismologies IPGP, T24, Case 89, 4, Place jussieu, F75252 Paris Cedex 05, France. e-mail: mangeney@ipgp.jussieu.fr

potential and by looking for polynomial solutions for this potential. The derivation, although not straightforward, do not require complicated mathematics.

This analytical solution, which is not of any special physical interest, is developed to provide a test for the numerical ice sheet flow models and, here, we shall compare it with the results obtained with the model developed by Mangeney, Califano, and Castelnau (1996) and Mangeney, Califano, and Hutter (1997). In this model, the equations of the steady ice flow with free surface under isothermal conditions are solved—in particular, the complete set of mechanical equations have been solved numerically. The analytical solution shown here has been very useful to test all terms involved in the mechanical equations and to test the Precision and the convergence of the model.

First, we present the system of equations, then we develop the analytical solution, and finally, we compare the results of the numerical ice sheet flow model of Mangeney, Califano, and Hutter (1997) with this analytical solution.

EQUATIONS

The two-dimensional flow of a viscous, incompressible material is described by mass and momentum conservation equations and a flow law. The material is supposed here to be isothermal so that the heat equation is not taken into account. All equations and variables are non-dimensionalized using typical depth and velocity (Mangeney, Califano, and Castelnau, 1996). Mass and momentum balance equations can be written as

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{S}' = \nabla p' \tag{2}$$

where $\mathbf{u} = (u_x, u_z)$ is the velocity vector, $\mathbf{S}'$ the deviatoric stress tensor, and p' the dynamic pressure calculated from the fluid pressure $p(p' = p - z)$. Here x and z denotes the horizontal and vertical axis, respectively (Fig. 1). These balance laws are supplemented by the constitutive relation for a viscous fluid:

$$\mathbf{S}' = \mathbf{M} \cdot \mathbf{D} \tag{3}$$

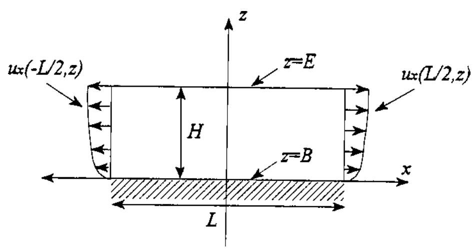

Figure 1. Coordinate system.

where **M** and **D** denote the fourth- and second-rank viscosity and strain rate tensors, respectively. For an isotropic material, with Newtonian behavior and vertically constant viscosity, the viscosity tensor **M** reduces to a constant viscosity times the fourth rank unit tensor, and the flow law (3) reads

$$\mathbf{S}' = \eta \mathbf{D} \tag{4}$$

We consider here the simple case of a steady-state plane flow of an isotropic material with constant viscosity in a 2D slab geometry with a free surface $z = E(x)$ over the base $z = B(x)$ (Fig. 1). As boundary conditions a prescribed velocity is imposed at the base $(u(x, B) = u_G(x))$ while the steady state kinematic condition at the free surface is given by

$$u_x \frac{\partial E}{\partial x} - u_z = a(x), \quad \text{at } z = E(x) \tag{5}$$

where a is a source term. For example, for ice sheet flow, a corresponds to the accumulation rate at the surface. The stress free top surface condition reads

$$\sigma \cdot \mathbf{n}_s - p_{\text{atm}} \mathbf{n}_s = 0, \quad \text{at } z = E(x) \tag{6}$$

where P_{atm} is the atmospheric pressure and $\mathbf{n}_s$ the exterior unit normal vector at the surface. Here σ is the stress tensor defined as $\sigma = \mathbf{S}' - p\mathbf{I}$, where $\mathbf{I}$ is the identity tensor. The flow is assumed to be symmetrical with respect to the axis $x = 0[u(0, z) = 0]$, typical of the situation of the flow around a dome. Therefore, identical boundary conditions for the horizontal velocity are imposed on the left- and right-hand side of the domain.

ANALYTICAL SOLUTION

We impose here two particular conditions: the bedrock and the free surface are assumed to be flat so that $E(x) = 1$ and $B(x) = 0$ (Fig. 1). Let us introduce the velocity potential:

$$\begin{aligned} u_x &= \partial_z \psi \\ u_z &= \partial_x \psi \end{aligned} \tag{7}$$

where $\partial_i f$ represents the ith derivative of the field f, so that the momentum Equation (2) can be written as

$$-\eta \partial_z (\Delta \psi) = \partial_x p \tag{8a}$$

$$\eta \partial_x (\Delta \psi) = \partial_z p \tag{8b}$$

By differentiating (8a) with respect to z and (8b) with respect to x and then subtracting these two equations, we obtain the biharmonic Poisson equation for

$$\Delta^2\psi = 0 \tag{9}$$

Let us now look for a polynomial solution of degree 4.

We shall impose here some particular boundary conditions in order to compare with the results of the numerical model described below (however, several other boundary conditions can be imposed for other types of viscous, incompressible flow). Because ψ is of order 4, velocities are of order 3—for example, the horizontal velocity at the base ($z = B = 0$) is

$$u_x = -\partial_z\psi = g_1 + g_2x + g_3x^2 + g_4x^3 \tag{10}$$

where g_i are the coefficients. Boundary conditions are, at the base ($z = B = 0$),

$$u_z = \partial_x\psi = 0 \tag{11}$$

and at the surface ($z = E$), the stress-free surface condition (6) for a flat surface,

$$\begin{aligned} p &= p_{\mathrm{atm}} - 2\eta\partial_x u_x \\ \tau_{xz} &= \eta(\partial_z u_x + \partial_x u_z) = 0 \end{aligned} \tag{12}$$

where τ_{xz} is the shear stress. This gives two conditions at $z = E(x)$:

$$\begin{aligned} \partial_x^2\psi - \partial_z^2\psi &= 0 \\ -2\eta\partial_{xz}^2\psi &= p_{\mathrm{atm}} - p \end{aligned} \tag{13}$$

and the polynomials of lowest degree for ψ and p, which allow a solution of these equations:

$$\begin{aligned} \psi &= g_0 - g_1z - g_2xz + g_3(-4Ez - x^2z + z^3) + g_4(-12Exz^2 - x^3z + 3xz^3) \\ p &= p_{\mathrm{atm}} - 2\eta(g_2 + 2g_3x + 3g_4x^2) + 6\eta g_4(z^2 - 4Ez - 2E^2) + E \end{aligned} \tag{14}$$

where g_0 is an arbitrary constant. We then obtain the velocities u_x and u_z:

$$\begin{aligned} u_x &= -\partial_z\psi = g_1 + g_2x + g_3(8Ez + x^2 - 3z^2) + g_4(24Exz + x^3 - 9xz^2) \\ u_z &= \partial_x\psi = -g_2z - 2g_3xz - 3g_4(4Ez^2 + x^2z - z^3) \end{aligned} \tag{15}$$

In the case of a flow symmetric with respect to $x = 0$, and having horizontal velocity equal to zero at this point (for example, the case of an ice sheet divide), coefficients g_i have to satisfy

$$g_1 = g_3 = 0 \tag{16}$$

By choosing arbitrarily $g_2 = 0$, the analytical solution for the pressure and velocity fields reduces then to

$$u_x = -\partial_z \psi = g_4(24Exz + x^3 - 9xz^2) \tag{17a}$$

$$u_z = \partial_x \psi = -3g_4(4Ez^2 + x^2 z - z^3) \tag{17b}$$

$$p = p_{\text{atm}} - 6\eta g_4(x^2 - z^2 + 4Ez + 2E^2) + E \tag{17c}$$

Different choice of g_2 would lead to another analytical solution.

PRECISION AND PERFORMANCE OF THE NUMERICAL MODEL

We have used the analytical solution presented here to test the numerical ice sheet flow model developed by Mangeney, Califano, and Hutter (1997). The system of Equations (1), (2), and (4) with boundary conditions (5) and (6) and the no-slip boundary condition $[u_z(x, 0) = 0]$ is solved numerically by reintroducing the time derivative of the velocity in the momentum equation (Mangeney, Califano, and Hutter, 1997) and by iterating in time t until convergence. The fictitious temporal problem,

$$\begin{aligned} \partial_t \mathbf{u} &= T\mathbf{u} - \nabla p \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{18}$$

where T is a second-order differential operator, is solved by using a finite difference method, semi-implicit in time. One point should be stressed here. The form (17a) imposes a given vertical profile of the horizontal velocity at the boundaries $x = \pm L/2$ of the numerical domain. Therefore numerical runs were performed by imposing this particular boundary condition at the left- and right-hand side of the numerical domain. In a similar manner the accumulation rate a that is imposed is obtained from Equation (5) and corresponds to the analytical solution for $u_z(x, E)$,

$$a = 3g_4 E(x^2 + 3E^2) \tag{19}$$

Starting with a zero velocity field or with an arbitrary velocity field, the time relaxation method converges more or less rapidly toward the stationnary state, depending on the value of the time step used in the numerical method. The deviation from the analytical solution depends on the time step. One of the interesting things of this comparison has been to understand the efficiency of the time iteration method to obtain the final stationary equilibrium. The basic problems are (a) the number of time steps to achieve convergence, (b) how far the state obtained at convergence is from the actual solution, here the analytical solution, and finally (c) does the numerical precision correspond to a second-order scheme?

We have tested the convergence toward the analytical solution for various time steps Δt and for various spatial steps Δx and Δz. The tests have been made using a parameter proportional to the time step and depending on the diagonal part of the operator T, satisfying $0 < \theta < 1$. The expression of θ is given in Figure 3 of Mangeney, Califano, and Hutter (1997) and describes the degree of time implicitiness. For $\theta = 0$, the time advance is explicit while for θ close to 1 the diagonal part of the second order differential operator T is fully implicit.

For θ varying from 0.1 to 0.9 and for 11×11 points, a precision of less than 10^{-2} for the relative error on the velocity field is obtained (Fig. 2). For θ ranging in this interval and $\theta < 0.5$, increasing the time step helps to decrease the time necessary to convergence and significantly improves the precision of the solution with respect to the analytical solution (Fig. 2). For $\theta > 0.5$, increasing the time deteriorates the solution. For $0.9 > \theta > 0.15$ and for 11×11 points, the numerical solution converges to the analytical solution after 400 iterations (1 hr CPU time) with a maximum difference lower than 10^{-4} between the velocities obtained at two successive iterations (Figs. 2 and 3). When the numerical model

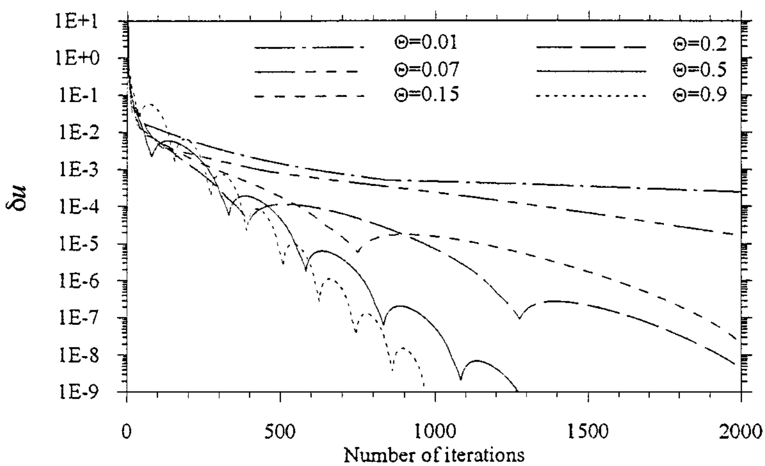

Figure 2. Convergence toward the analytical solution of the numerical solution obtained for different time steps θ for (11×11) points. A, Maximum difference between the velocity of the numerical solution and that of the analytical solution $[\Delta u = \max(u_i - u_{i\text{anal}}), i = x, z]$ divided by the mean velocity (u_m); B, mean divergence of the velocities (div u) divided by the mean velocity gradient (du_m).

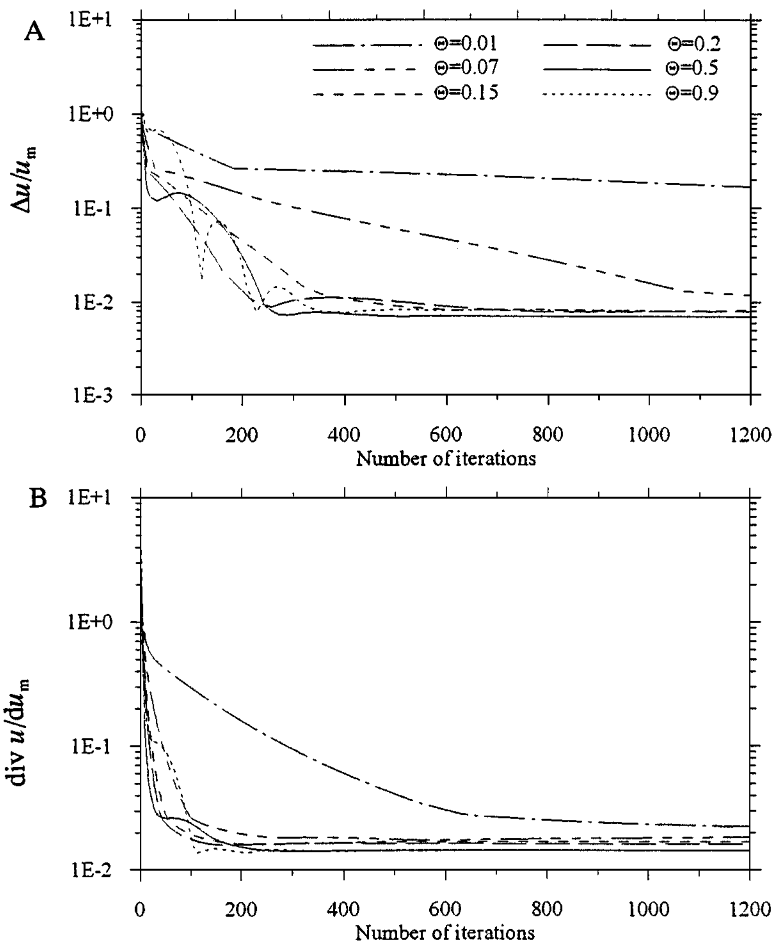

Figure 3. Maximum difference between the velocities obtained at two successive iterations $[\delta u = \max(u_i^{n+1} - u_i^n), i = x, z]$ vs. the number of iterations for different time steps θ for (11×11) points.

is used with evolving free surface (Mangeney, Califano, and Hutter, 1997), an analytical solution is not available. Convergence is then achieved by imposing the maximum difference between velocities obtained at two successive iterations (δu) lower than a certain value. Comparison with the analytical solution allows one to assess the precision of the numerical solution for a given value of δu (Figs. 2 and 3). For 21×21 points (Fig. 4), the time necessary to convergence is significantly increased (6000 iterations). The precision of the calculation increases until a threshold for θ is reached, which depends on the number of points. This is

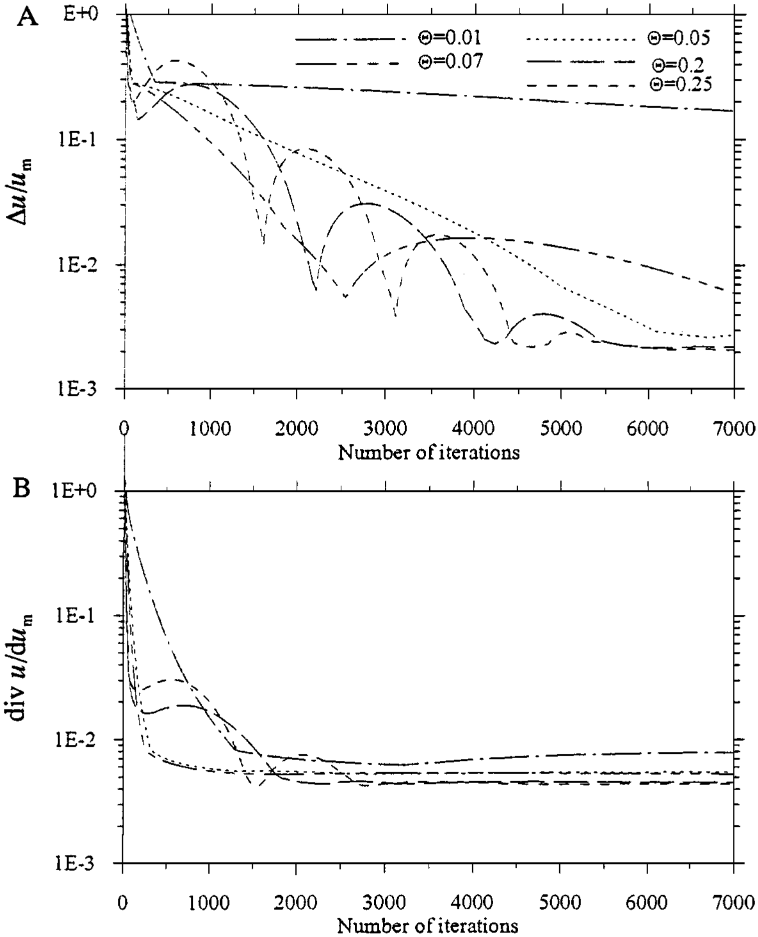

Figure 4. Convergence of the numerical solution towards the analytical solution obtained for different time steps θ for (21×21) points. A, Maximum difference between the velocity of the numerical solution and that of the analytical solution $[\Delta u = \max(u_i - u_{i\,\mathrm{anal}}), i = x, z]$ divided by the mean velocity (u_m); B, mean divergence of the velocities divided by the mean velocity gradient (du_m) vs. the number of iteration n.

due to the behavior of the eigen values of the operator T. It seems that the threshold for θ decrease with the number of points. This comparison allows us to check how this method of artificial relaxation converges to the actual stationary solution. Note that some of the curves indicating the rate of convergence (Figs. 2–4) appear to periodically oscillate for certain values of θ. It is typical of these numerical methods (Sotiropoulos and Abdallah, 1991).

Figure 2 and 4 show that the difference with respect to the analytical solution and the error on the velocities divergence are approximately divided by 4 when the number of points is doubled. The precision of the numerical code appears to be of order 2. The error on the analytical solution and on the velocity divergence is of the order of the discretization error: for 11×11 points the discretization error, proportional to the square of the space step, is of order 10^{-2}, for 21×21 points it is of order 2.5×10^{-3}.

CONCLUSION

We have presented here an analytical solution for the linearly viscous Stokes equations allowing to test two-dimensional ice sheet flow models. We have used it to estimate the performance and precision of the model described in Mangeney, Califano, and Hutter (1997) which appears to be of order 2 in space. The comparison of this 2D ice sheet flow model with this analytical solution has allowed to test all the terms involved in the mechanical equations. This solution may be very useful to test similar types of models.

ACKNOWLEDGMENTS

This work was supported by the PNEDC (Programme National d'Etude du Climat), the Environment program of the CCE (in France) aids the EISMINT program (European Science Foundation). I am grateful to F. Califano for fruitful discussions and to D. Cohen for interesting comments.

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