Reply to Comment by E. Faccioli and A. Quateroni on "The Spectral Element Method: An Efficient Tool to Simulate the Seismic Response of 2D and 3D Geological Structures" by D. Komatitsch and J.P. Vilotte

by D. Komatitsch and J.P. Vilotte

We have read with interest the recent paper of Faccioli *et al.* (1997), published in the *Journal of Seismology.* This is indeed an illustration of the fact that two parallel and independent research efforts can sometimes partially converge. Not surprisingly, both groups have used similar 2D test-cases to validate their method and assess the related accuracy, since only very few test-cases have known analytical or semi-analytical solutions. Among these, Lamb's and Garvin's problems are probably the most popular, and have been used for instance in Komatitsch and Vilotte (1996) and Komatitsch (1997). Unfortunately, such a direct comparison to known solutions is more difficult in 3D since very few reference problems have been published. Faccioli et al. studied a relatively simple 3D problem in their paper. It would be of interest to compare our results for the realistic 3D propagation problems (Komatitsch and Vilotte, 1996) with their method.

The Spectral Element Method, including the formulation based on Legendre polynomials and Gauss-Lobatto Legendre quadrature, has become a classical high-order variational formulation in several fields of numerical modeling and that the related literature is in rapid expansion. Some of the major recent contributions that inspired our own work: Patera (1984), Rønquist and Patera (1987), Fisher *et al.* (1988), Rønquist and Patera (1988), Maday and Patera (1989), Bernardi and Maday (1992), Maday *et al.* (1993), Azaiez *et al.* (1995), Fisher and Rønquist (1994), Ho and Rønquist (1994). The Spectral Element Method has now become a natural high-order extension of the Finite Element Method, and is closely related to its original $h-p$ version. The potential accuracy of the Spectral Element method for wave propagation problems was demonstrated as early as 1991 by Seriani and Priolo (1991). Using a method based on Chebyshev polynomials and Gauss-Lobatto-Chebyshev quadrature, they were able to solve several difficult problems (Seriani *et al.* (1992), Seriani and Priolo (1994), Priolo *et al.* (1994)), even though their method was probably not optimal in terms of computational efficiency due to the specific choice made for the set of algebraic polynomials and collocation points. Recently, Cohen *et al.* (1993) and Tordjman (1995) also introduced a Spectral Element Method for acoustic wave propagation and derived some interesting analytical results regarding the related dispersion properties.

Our contribution is therefore part of a continuously evolving effort to use high-order variational formulations for solving problems involving elastic wave propagation. A large part of our results, including a detailed analysis of Garvin's problem and several 3D examples, was presented at the 1996 AGU fall meeting (Komatitsch and Vilotte, 1996) as well as in a Ph.D. Thesis Komatitsch (1997). The paper of Faccioli *et al.* (1997) came out while our own paper was in press in BSSA.

Regarding the two rather technical points brought up for attention by Faccioli and Quarteroni, we shall answer briefly for the sake of the reader's interest:

The time-stepping scheme used in our work, which is clearly stated both in the abstract and in the body of our paper, is indeed a predictor-multicorrector scheme that is constructed to be compatible with classical Newmark methods. Such a formulation is rather standard and can be found in classical textbooks (for instance, Hughes 1987, chap. 9). This predictor-multicorrector formulation leads to improved characteristics and to an efficient parallel implementation due to the explicit nature of the predictor phase. This is of great importance for the computational efficiency of the method and takes advantage of the inherent diagonal structure of the mass matrix. In our paper, we present an improved version of this scheme written in conservation form, as originally proposed by Simo and Tarnow (1992), Simo *et al.* (1992) and Gonzales and Simo (1996). In such a formulation, the choice $\alpha = \beta = 0.5$ and $\gamma = 1$ defines a second-order acceleration-independent one-leg multistep method that has the property of exact momentum conservation and defines a symplectic scheme for nonlinear elastodynamics. It does not reduce to a simple explicit predictor-corrector scheme. This is of importance in some circumstances, for instance when dealing with absorbing boundary conditions, or with attenuation (although attenuation was not considered in our paper). Classical implicit schemes are far too expensive for realistic applications and do not lead to significant improvement in wave propagation problems. Therefore, our scheme is conditionally stable and the Courant stability condition is a conservative condition. However, in most circumstances the key problem is not the stability condition, but the long-term dissipativity of the time-stepping algorithms and the related dispersion. Faccioli and Quarteroni therefore indirectly raise an interesting point. The main improvement upon which the Spectral Element Method is based is inherent to the spatial approximation and derives from the good approximation properties of the polynomial expansion, in conjunction with the accuracy associated with the Gauss-Lobatto Legendre quadrature and interpolation. In contrast, the time discretization is only second-order accurate. In some cases, improved schemes may be needed, for instance using a high-order time-stepping that retains the computational efficiency, as proposed recently by Tarnow and Simo (1994), leading to a fourth-order scheme with no significant overhead.

Faccioli and Quarteroni also raise the issue of defining a minimum number of points per wavelength. This is not an easy problem at all, particularly for the Spectral Element Method which is based, at the element level, on a local deformed non-evenly spaced grid. To our knowledge, no fully-general study of this problem is available yet. This issue has been partially addressed, in the acoustic case, by Cohen *et al.* (1993) and Tordjman (1995). Otherwise, such an estimation relies only on numerical results based on simple 1D and 2D test-problems, such as those of Seriani and Priolo (1994). The small discrepancy between Faccioli *et al.* (1997) and our work is not significant and simply comes from a slightly different way of defining the averaged condition. Our value of the threshold G is in agreement with Seriani and Priolo (1994). More details can be found in Komatitsch (1997, pp 71-74). In realistic geophysical problems, with complicated geometries and material heterogeneities, such an estimation is not obvious and an analytical or empirical a-priori condition is very difficult to define. It is worth to mention here the related work of Maday and Rønquist (1990) in fluid dynamics.

Finally, we would like to point out that the results obtained in our paper for the diffraction of elastic waves by a steep 3D topography clearly illustrate the efficiency of the Spectral Element Method both in terms of its accuracy and of its ability to handle complicated 3D structures. This can easily be assessed when comparing with the simpler 3D-test problem presented in

Faccioli *et al.* (1997). We are however strongly convinced that the method can still be improved by adding more geometrical flexibility such as the use of non-conforming discretization.

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